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PATH AND CYCLE DECOMPOSITIONS OF DENSE GRAPHS

ANTÓNIO GIRÃO, BERTILLE GRANET, DANIELA KÜHN, AND DERYK OSTHUS

ABSTRACT. We make progress on three long standing conjectures from the 1960s about path and cycle decompositions of graphs. Gallai conjectured that any connected graph on n vertices can be decomposed into at most $\lceil \frac{n}{2} \rceil$ paths, while a conjecture of Hajós states that any Eulerian graph on n vertices can be decomposed into at most $\lfloor \frac{n-1}{2} \rfloor$ cycles. The Erdős-Gallai conjecture states that any graph on n vertices can be decomposed into $O(n)$ cycles and edges.

We show that if G is a sufficiently large graph on n vertices with linear minimum degree, then the following hold.

- (i) G can be decomposed into at most $\frac{n}{2} + o(n)$ paths.
- (ii) If G is Eulerian, then it can be decomposed into at most $\frac{n}{2} + o(n)$ cycles.
- (iii) G can be decomposed into at most $\frac{3n}{2} + o(n)$ cycles and edges.

If in addition G satisfies a weak expansion property, we asymptotically determine the required number of paths/cycles for each such G .

- (iv) G can be decomposed into $\max \left\{ \frac{\text{odd}(G)}{2}, \frac{\Delta(G)}{2} \right\} + o(n)$ paths, where $\text{odd}(G)$ is the number of odd-degree vertices of G .
- (v) If G is Eulerian, then it can be decomposed into $\frac{\Delta(G)}{2} + o(n)$ cycles.

All bounds in (i)–(v) are asymptotically best possible.

1. INTRODUCTION

1.1. Background. Graph decomposition is a central field of graph theory, which encompasses some of the oldest and most famous problems in combinatorics. For example, the decomposition of complete graphs into Hamilton cycles or Hamilton paths was attributed to Walecki and dates back to 1883 [30] (see [2] for a description in English of Walecki's construction). Extensive research has also been done on decompositions of graphs into (not necessarily Hamiltonian) paths and/or cycles. One of the most famous results in this area is due to Lovász.

Theorem 1.1 ([29]). *Let G be a graph on n vertices. Then G can be decomposed into at most $\lceil \frac{n}{2} \rceil$ paths and cycles.*

We observe that this result is sharp. Indeed, a vertex of odd degree in a graph G must be the endpoint of at least one path in a path and cycle decomposition of G . Thus, n -vertex graphs with at most one vertex of even degree cannot be decomposed into fewer than $\lceil \frac{n}{2} \rceil$ paths and cycles.

The result of Lovász was inspired by the following conjecture of Gallai (see [29]).

Conjecture 1.2 (Gallai). *Any connected graph on n vertices can be decomposed into at most $\lceil \frac{n}{2} \rceil$ paths.*

Complete graphs show that the conjecture of Gallai would be best possible. Lovász [29] observed that Theorem 1.1 implies that any graph can be decomposed into at most $n - 1$

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paths. This was later improved by Donald [12] who showed that $\lfloor \frac{3n}{4} \rfloor$ paths are sufficient. It was subsequently shown by Dean and Kouider [11] and independently by Yan [33] that $\lfloor \frac{2n}{3} \rfloor$ paths suffice. The covering version of Gallai's conjecture (where the paths are not necessarily edge-disjoint) was solved by Fan [15].

Although the conjecture of Gallai remains open, it has been verified for several classes of graphs. We direct the readers to [5, 7, 17, 21, 24, 29, 32] for some of these results.

The analogous problem for cycle decompositions was posed by Hajós (see [29]). We note that the original problem suggested by Hajós asked for a decomposition of Eulerian n -vertex graphs into at most $\lfloor \frac{n}{2} \rfloor$ cycles, but Dean [10] observed that this is equivalent to the following.

Conjecture 1.3 (Hajós). *Any Eulerian graph on n vertices can be decomposed into at most $\lfloor \frac{n-1}{2} \rfloor$ cycles.*

Eulerian graphs with maximum degree $n - 1$ demonstrate that the conjecture of Hajós would be best possible. Conjecture 1.3 has only been verified for specific classes of graphs. See [18] for some of these results. Again, the analogous covering problem was resolved by Fan [16].

Jackson [22] conjectured the analogue of Conjecture 1.3 for Eulerian oriented graphs. However, Dean [10] observed that this conjecture is false and conjectured instead that any Eulerian oriented graph on n vertices can be decomposed into $\lfloor \frac{2n}{3} \rfloor$ dicycles and any Eulerian digraph on $n > 1$ vertices can be decomposed into $\frac{8n}{3} - 3$ dicycles.

Very little progress has been made on Conjecture 1.3 for general graphs. In particular, the related problem of decomposing Eulerian graphs into $O(n)$ cycles is still open and is equivalent to a problem posed in [14] which is known as the Erdős-Gallai conjecture (see [13]).

Conjecture 1.4 (Erdős-Gallai). *Any graph on n vertices can be decomposed into $O(n)$ cycles and edges.*

Observe that given any n -vertex graph G , by repeatedly removing cycles until no longer possible, we obtain a forest F such that $G \setminus F$ is Eulerian. Since this forest contains at most $n - 1$ edges, the problem of decomposing graphs into $O(n)$ cycles and edges reduces to decomposing Eulerian graphs into $O(n)$ cycles. Conversely, given a decomposition of an Eulerian graph G into $O(n)$ cycles and edges, one can easily obtain a decomposition of G into $O(n)$ cycles. Thus, Conjecture 1.4 is equivalent to the problem of decomposing Eulerian graphs into $O(n)$ cycles. Also observe that Conjecture 1.3 would imply that any graph can be decomposed into at most $\frac{3(n-1)}{2}$ cycles and edges. Thus, the Erdős-Gallai conjecture holds for all classes of graphs for which Conjecture 1.3 has been verified. Additionally, the Erdős-Gallai conjecture was verified for graphs of linear minimum degree by Conlon, Fox, and Sudakov [8]. More precisely, they showed the following.

Theorem 1.5 ([8]). *For any $\alpha > 0$, if G is a graph on n vertices with minimum degree $\delta(G) \geq \alpha n$, then G can be decomposed into $O(\alpha^{-12}n)$ cycles and edges.*

Conjecture 1.4 remains open for general graphs, while the covering version was proved by Pyber [31].

It is not hard to show that $\frac{n \log(n)}{2} + O(n)$ cycles and edges are sufficient to decompose any graph (see [13]). An example of Erdős [13] shows that at least $(\frac{3}{2} - o(1))n$ cycles and edges are necessary for some graphs. It was recently shown in [8] that $O(n \log \log n)$ cycles and edges are sufficient, and this is currently the best known result for general graphs. More precisely, they proved the following.

Theorem 1.6 ([8]). *Let G be a graph on n vertices with average degree d . Then G can be decomposed into $O(n \log \log d)$ cycles and edges.*

Progress on Conjectures 1.2 and 1.4 was also made for random graphs. As proved in [8], for any edge probability $p := p(n)$, the binomial random graph $G(n, p)$ satisfies Conjecture 1.4 asymptotically almost surely. More details about decompositions of random graphs into cycles and edges can be found in [26], where Korándi, Krivelevich and Sudakov provided an asymptotically tight result for a large range of edge probabilities $p(n)$.

For constant edge probability $0 < p < 1$, Glock, Kühn, and Osthus [20] strengthened the bounds of [8, 26] to obtain precise results for decompositions of $G(n, p)$ into paths or cycles and into matchings. In fact, they obtained their results for quasirandom graphs. More precisely, they used the following notion of quasirandomness. An n -vertex graph G is *lower*-(ε, p)-*regular* if for any disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$, we have $e_G(S, T) \geq (p - \varepsilon)|S||T|$. Given a graph G , we denote by $\text{odd}(G)$ the number of odd-degree vertices of G .

Theorem 1.7 ([20]). *For any $0 < p < 1$, there exist $\varepsilon, \eta, n_0 > 0$ such that for any $n \geq n_0$ the following hold. Let G be a lower-(ε, p)-regular graph on n vertices with $\Delta(G) - \delta(G) \leq \eta n$. Then,*

- (i) G can be decomposed into $\max \left\{ \frac{\text{odd}(G)}{2}, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}$ paths, and
- (ii) if G is Eulerian, then it can be decomposed into $\frac{\Delta(G)}{2}$ cycles.

These bounds are best possible for each G , but do not hold in general (some examples can be found in Section 6).

Bienia and Meyniel [3] conjectured the analogue of Conjecture 1.4 for Eulerian digraphs.

Conjecture 1.8 (Bienia and Meyniel). *There exists $\alpha \in \mathbb{R}$ such that any Eulerian digraph on n vertices can be decomposed into at most αn dicycles.*

As mentioned in [3, 10], unions of complete symmetric digraphs K_4^* which are all sharing a common vertex show that, if Conjecture 1.8 is true, then $\alpha \geq \frac{4}{3}$. Conjecture 1.8 is also discussed in [4]. It is still open but some progress was recently made by Knierim, Larcher, Martinsson, and Noever [25].

Theorem 1.9 ([25]). *Let D be an Eulerian digraph on n vertices and with maximum degree Δ . Then D can be decomposed into $O(n \log \Delta)$ dicycles.*

1.2. New results. First, we prove approximate versions of Conjectures 1.2 and 1.3 for sufficiently large graphs of linear minimum degree (see Theorems 1.10(i) and 1.10(ii)). Theorem 1.10(ii) easily implies Theorem 1.10(iii), which improves Theorem 1.5 and gives (asymptotically) the best possible constant.

Theorem 1.10. *For any $\alpha, \delta > 0$, there exists n_0 such that if G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq \alpha n$, then the following hold.*

- (i) G can be decomposed into at most $\frac{n}{2} + \delta n$ paths.
- (ii) If G is Eulerian, then it can be decomposed into at most $\frac{n}{2} + \delta n$ cycles.
- (iii) G can be decomposed into at most $\frac{3n}{2} + \delta n$ cycles and edges.

Secondly, we prove approximate versions of the bounds in Theorem 1.7 for sufficiently large graphs with linear minimum degree which satisfy a weak version of quasirandomness. More precisely, we say an n -vertex graph G is *weakly*-(ε, p)-*quasirandom* if for any partition $A \cup B$ of $V(G)$ with $|A|, |B| \geq \varepsilon n$ we have $e_G(A, B) \geq p|A||B|$. This notion of weak quasirandomness implies that the reduced graph obtained after applying the regularity lemma to a dense graph is connected. This is the only property required to obtain the bounds in the following theorem.

Theorem 1.11. *For any $\alpha, \delta, p > 0$, there exists n_0 such that if G is a weakly-($\frac{\alpha}{2}, p$)-quasirandom graph on $n \geq n_0$ vertices with $\delta(G) \geq \alpha n$, then the following hold.*

- (i) G can be decomposed into at most $\max \left\{ \frac{\text{odd}(G)}{2}, \frac{\Delta(G)}{2} \right\} + \delta n$ paths.
- (ii) If G is Eulerian, then it can be decomposed into at most $\frac{\Delta(G)}{2} + \delta n$ cycles.

In particular, the following holds.

Corollary 1.12. *For any $\delta, \varepsilon > 0$, there exists n_0 such that if G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq \frac{n}{2} + \varepsilon n$ then the following hold.*

- (i) G can be decomposed into at most $\max \left\{ \frac{\text{odd}(G)}{2}, \frac{\Delta(G)}{2} \right\} + \delta n$ paths.

(ii) If G is Eulerian, then it can be decomposed into at most $\frac{\Delta(G)}{2} + \delta n$ cycles.

Note that, if in addition G is regular, then the error terms of εn and δn can be removed in Corollary 1.12(ii), see [9].

The next result shows that one can drop the linear minimum degree condition in Theorem 1.11(i) if the quasirandomness covers a larger range of partition class sizes.

Theorem 1.13. *For any $p, \delta > 0$, there exist $\varepsilon, n_0 > 0$ such that the following holds. If G is a weakly- (ε, p) -quasirandom graph on $n \geq n_0$ vertices, then G can be decomposed into at most $\max \left\{ \frac{\text{odd}(G)}{2}, \frac{\Delta(G)}{2} \right\} + \delta n$ paths.*

For Theorem 1.10, the linear minimum degree condition is likely to be an artefact of our proof. On the other hand, in Section 6, we will give some examples to show that neither the linear minimum degree condition (or even the stronger assumption of linear connectivity), nor the weakly- $(\frac{\alpha}{2}, p)$ -quasirandom property is sufficient on its own to obtain the bounds in Theorem 1.11. However, Theorem 1.13 shows that, in the case of path decompositions, the linear minimum degree condition can be dropped if we assume G to be weakly- (ε, p) -quasirandom for a sufficiently small constant $\varepsilon > 0$. Surprisingly, it turns out that the Erdős-Gallai conjecture is equivalent to the following analogue of Theorem 1.13 for cycle decompositions of Eulerian graphs (see Proposition 6.3).

Conjecture 1.14. *For any $\delta, p > 0$, there exist $\varepsilon, n_0 > 0$ such that the following holds. If G is an Eulerian weakly- (ε, p) -quasirandom graph on $n \geq n_0$ vertices, then G can be decomposed into at most $\frac{\Delta(G)}{2} + \delta n$ cycles.*

We can prove Conjecture 1.14 if weak- (ε, p) -quasirandomness is replaced by weak- $(\frac{\varepsilon}{\log \log n}, p)$ -quasirandomness (see Proposition 6.4).

We note that Theorems 1.11 and 1.13 differ from Theorem 1.7 in the following way. Firstly, we have no restriction on the difference between the maximum and minimum degree. Secondly, weak- (ε, p) -quasirandomness is a significantly weaker property than lower- (ε, p) -regularity. Moreover, the ε -parameter in Theorem 1.7 is much smaller than the p -parameter. We do not require this in Theorem 1.11, and while this is necessary in Theorem 1.13, there we do not require the minimum degree to be linear. On the other hand, Theorems 1.11 and 1.13 have an additional $o(n)$ term in the number of paths/cycles compared to Theorem 1.7.

Finally, we observe that the following is immediately implied by Corollary 1.12.

Corollary 1.15. *For any $\varepsilon > 0$, there exists n_0 such that the conjecture of Hajós is true for all Eulerian graphs G on $n \geq n_0$ vertices with $\delta(G) \geq \frac{n}{2} + \varepsilon n$ and $\Delta(G) \leq n - \varepsilon n$.*

We remark that by Theorem 1.11, Corollary 1.15 holds more generally for sufficiently large weakly-quasirandom graphs with maximum degree bounded away from n .

A key tool in our proofs will be the main technical result of [27], which generates a Hamilton decomposition of a graph satisfying certain robust expansion properties (see Section 4.4 for the statement). This was developed originally in [27] to give a proof of Kelly's conjecture (which states that every large regular tournament has a Hamilton decomposition), and applied e.g. in [9] to prove the 1-factorisation conjecture (see also [28] for some early applications).

1.3. Organisation of the paper. The paper is organised as follows. We start by providing a proof overview of our main theorems in Section 2. Notation and probabilistic tools are introduced in Section 3, and preliminary results are collected in Section 4. Theorems 1.10(i), 1.10(ii), 1.11, and 1.13 are proved in Section 5. Finally, we derive Theorem 1.10(iii) and make some concluding remarks in Section 6.

2. PROOF OVERVIEW OF THE MAIN THEOREMS

The proofs of Theorems 1.10(i), 1.10(ii), 1.11 and 1.13 follow a similar strategy, and so, for simplicity, we only sketch the proof of Theorem 1.10(ii).

Fix additional constants $\varepsilon, \zeta, \beta$, and n_0 such that $0 < \frac{1}{n_0} \ll \varepsilon \ll \zeta \ll \beta \ll \alpha, \delta \leq 1$. Let G be a graph on $n \geq n_0$ vertices with $\delta(G) \geq \alpha n$. We decompose G by repeatedly constructing cycles. For simplicity, whenever edges are used to form a cycle, they are implicitly deleted from the graph (so all the cycles constructed below are edge-disjoint, as desired). We obtain the bulk of our cycles in Step 3, all other cycles will contribute to the error term. In Step 3, we need to be very efficient (i.e. the average length of the cycles needs to be large), while there is room to spare in the other steps.

Step 1: Applying Szemerédi’s regularity lemma and setting aside some random subgraphs Γ and Γ' . We start by applying Szemerédi’s regularity lemma and a cleaning procedure similar to the one used to prove the degree form of the regularity lemma. We will thus obtain a subgraph $H \subseteq G$ of small maximum degree and a partition of $V(G)$ into clusters V_1, \dots, V_k and an exceptional set V_0 . Moreover, in each non-empty pair of clusters of $G \setminus H$, almost all vertices have degree close to the density of the pair, while the few other vertices are isolated. Moreover, in each pair, the vertices of positive degree span an ε -regular bipartite graph.

We also set aside two sparse edge-disjoint random spanning subgraphs $\Gamma, \Gamma' \subseteq G \setminus H$ such that, in Γ , each non-empty pair of clusters has density close to β , while in Γ' each such pair has density close to ζ . By Theorem 1.1 and by splitting clusters if necessary, we may assume that the reduced graph R' of Γ can be decomposed into at most $\frac{|R'|}{2} = \frac{k}{2}$ cycles of even length (this will be needed in Step 5). Let $G^* := G \setminus (H \cup \Gamma \cup \Gamma')$. Denote by G_{ij}^* the ε -regular (almost spanning) subgraph of the pair $G^*[V_i, V_j]$, and define Γ_{ij} similarly. Γ and Γ' will be used to tie together given sets of paths of G^* into cycles.

Step 2: Covering the edges of $G[V_0]$. Apply Theorem 1.1 to $G[V_0]$. The paths obtained are extended to paths with endpoints in $V(G) \setminus V_0$ and then closed into cycles using edges of Γ . Since V_0 is small, this results in only a few cycles and we can use edges of Γ sparingly so that its properties are not destroyed.

Step 3: Covering most of G^* with at most roughly $\frac{n}{2}$ cycles. The idea is to decompose the edges of G^* into paths and then link some of these paths together using the edges in $\Gamma \cup \Gamma'$ to form cycles. The bipartite graph $G^*[V_0, V(G) \setminus V_0]$ is decomposed into paths of length 2 with midpoints in V_0 , called *exceptional paths*, while ε -regular pairs G_{ij}^* are approximately decomposed into long but not spanning paths, so that a few vertices are set aside for tying up paths. We then use edges of $\Gamma \cup \Gamma'$ to link these paths into cycles. More precisely, we proceed as follows. Suppose first that the reduced graph R of G is connected. We construct an auxiliary reduced graph \hat{R} such that the multiplicity of the edges between V_i and V_j in \hat{R} is proportional to the density of corresponding pair G_{ij}^* of G^* . We optimally decompose \hat{R} into matchings. Given a matching M of \hat{R} , we form sets \mathcal{P} of paths consisting of exactly one path of G_{ij}^* for each $V_i V_j \in M$, and of exceptional paths which cover vertices of V_0 with highest degree. Since M is a matching of clusters and our non-exceptional paths do not span entire clusters, we can ensure that each set \mathcal{P} of paths obtained in this way consists of vertex-disjoint paths and does not span entire clusters. Thus, after this step, we still have some uncovered vertices, called *reservoir vertices*, which can be used to link the paths in each set \mathcal{P} into a cycle using edges of $\Gamma \cup \Gamma'$.

Since the edge multiplicity between two clusters in \hat{R} is proportional to the density of the corresponding pair of G^* and at each stage we cover exceptional vertices of highest degree, we obtain an upper bound of roughly $\frac{\Delta(G^*)}{2}$ cycles in total. In general, R may be disconnected and, by construction, $\Gamma \cup \Gamma'$ contains no edges between the different components of R . Thus, we cannot tie together paths from different components and we need to apply the above argument to each component of R separately. But, if a component of R contains n' vertices of G^* (say), then the subgraph of G^* induced by this component has maximum degree at most n' and we obtain at most roughly $\frac{n'}{2}$ cycles from that component. Thus, we get an upper bound of roughly $\frac{n}{2}$ cycles in total.

By alternating which vertices are used as reservoir vertices, we ensure that the leftover graph H' has small maximum degree. Moreover, we use edges of Γ sparingly so that the properties of Γ are maintained. Since the density ζ of Γ' is small, we can add the remaining edges of Γ' to H' without significantly increasing the maximum degree of H' .

We remark that in Step 2 it was possible to tie together paths using only Γ because we had some room to spare (in the sense that the number of cycles produced might be fairly large compared to the number of edges covered). But in Step 3, we need to use edges of both Γ and Γ' in order to be efficient and obtain the desired number of cycles. (The reason that using $\Gamma \cup \Gamma'$ is more efficient is that the reduced graph of $\Gamma \cup \Gamma'$ equals that of G^* . We cannot guarantee this property for Γ alone since for Step 4 the non-empty pairs Γ_{ij} of Γ need to be fairly dense.)

Step 4: Covering the leftovers. By construction, $H \cup H'$ has small maximum degree and so can be decomposed into few small matchings. We tie the edges of each matching into a cycle using edges of Γ . Once again, we make sure that the relevant properties of Γ are preserved.

Step 5: Fully decomposing Γ . It only remains to decompose (the remainder of) Γ . The

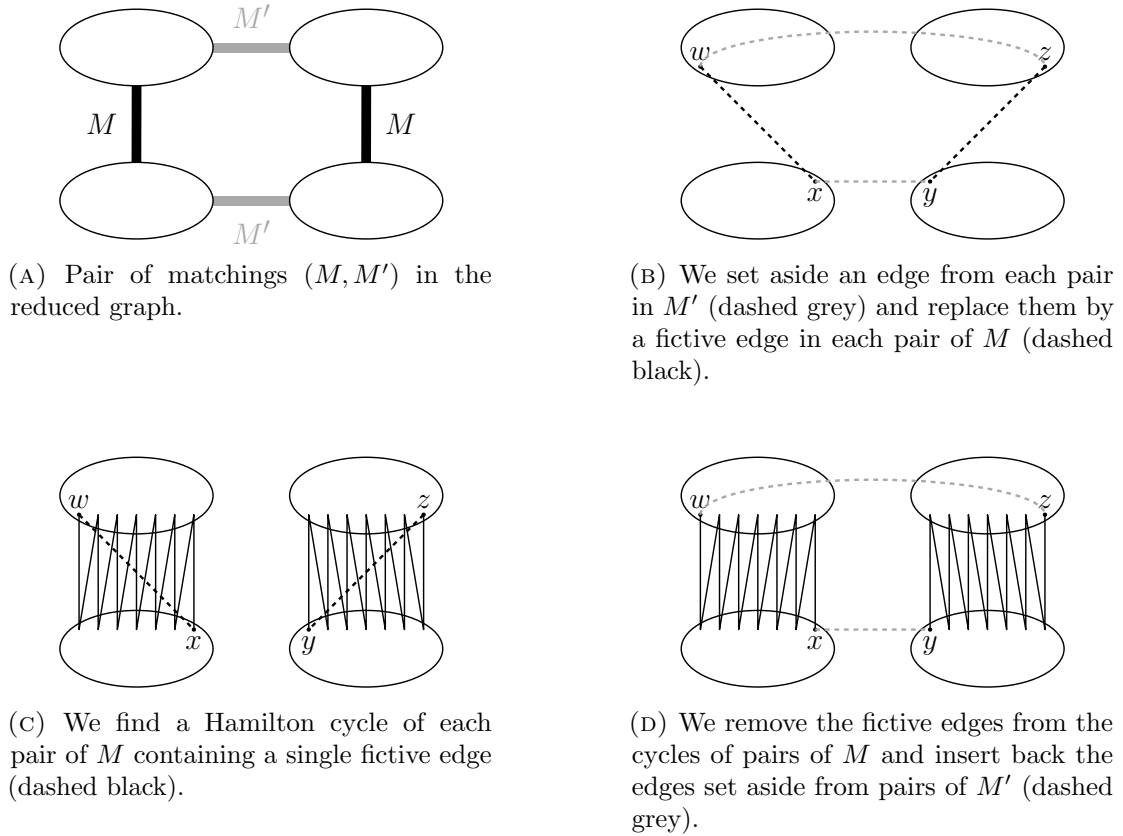


FIGURE 1. Construction of a cycle of Γ .

idea is to initially decompose the reduced graph of Γ into $\frac{k}{2}$ cycles of even length (as discussed in Step 1). For each such cycle C , the subgraph Γ_C of Γ corresponding to the blow-up of C is first approximately decomposed into Hamilton cycles of Γ_C that “wind around” C . The leftover is then decomposed using the main technical result of [27] as follows.

The cycle C is initially decomposed into a pair (M, M') of matchings. For each $V_i V_j \in M \cup M'$, we first set aside a small set \mathcal{E}_{ij} of edges of Γ_{ij} and then decompose the remaining edges into a set \mathcal{H}_{ij} of Hamilton paths. We make sure the set of endpoints of the paths in $\bigcup_{V_i V_j \in M} \mathcal{H}_{ij}$ equals the set of endpoints of the edges in $\bigcup_{V_i V_j \in M'} \mathcal{E}_{ij}$, and similarly for M and M' exchanged. Thus we can tie together a path of \mathcal{H}_{ij} for each $V_i V_j \in M$ using exactly one edge of $\mathcal{E}_{i'j'}$ for

each $V_i V_{j'} \in M'$. We proceed similarly to tie paths of $\bigcup_{V_i V_{j'} \in M'} \mathcal{H}_{ij}$ into cycles. We thus obtain a Hamilton decomposition of Γ_C .

In order to prescribe the endpoints of the Hamilton paths, we add some suitable edges to Γ_C , called *fictive edges*, and then actually find a Hamilton decomposition of each pair $\Gamma_{ij} \setminus \mathcal{E}_{ij}$ such that each cycle in the decomposition contains exactly one fictive edge (see Figure 1). Such decompositions are guaranteed by the “robust decomposition lemma” of [27]. Since by construction all pairs of Γ have density close to β , we obtain, in total, about $\frac{\beta n}{2} \ll \delta n$ cycles.

3. NOTATION, DEFINITIONS, AND PROBABILISTIC TOOLS

3.1. Notation. Let G be a graph. If $X \subseteq V(G)$ is a set of vertices of G we write $G[X]$ for the subgraph of G induced by X and $G - X$ for $G[V(G) \setminus X]$. Given a set $F \subseteq E(G)$ of edges of G , we write $G \setminus F$ for the graph obtained from G by deleting all edges in F . Similarly, given a subgraph $H \subseteq G$, we write $G \setminus H$ for $G \setminus E(H)$. If F is a set of non-edges of G , we write $G \cup F$ for the graph obtained from G by adding all edges in F . If G and H are edge-disjoint graphs we write $G \cup H$ for the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Assume G is a graph. For any $x \in V(G)$, we denote by $N_G(x)$ the set of neighbours of x and by $d_G(x)$ the degree of x in G . Given $x, y \in V(G)$, we define $d_G(x, y) := |N_G(x) \cap N_G(y)|$. The subscripts may be omitted if this is unambiguous. We say G is *Eulerian* if all its vertices have even degree. (Note that G is not necessarily connected.)

Given a graph G and $A, B \subseteq V(G)$, we write $e_G(A, B)$ for the number of edges of G which have an endpoint in A and an endpoint in B . If A, B are disjoint then we write $G[A, B]$ for the bipartite subgraph of G with vertex classes A and B and all edges of G with an endpoint in A and an endpoint in B .

Let \vec{G} be a digraph. Given vertices $x, y \in V(\vec{G})$, we write xy for the edge directed from x to y . The vertex x is called the *initial vertex* of xy and y the *final vertex* of xy . Given a vertex $x \in V(\vec{G})$, the *outneighbourhood* of x , denoted $N_{\vec{G}}^+(x)$, is the set of vertices y such that $xy \in E(\vec{G})$. Similarly, the *inneighbourhood* $N_{\vec{G}}^-(x)$ of a vertex $x \in V(\vec{G})$ is the set of vertices y such that $yx \in E(\vec{G})$. We say \vec{G} is r -regular if for any vertex $x \in V(\vec{G})$, we have $|N_{\vec{G}}^+(x)| = |N_{\vec{G}}^-(x)| = r$. For any $A, B \subseteq V(\vec{G})$, we write $e_{\vec{G}}(A, B)$ for the number of edges of \vec{G} whose initial vertex belongs to A and whose final vertex belongs to B . For any disjoint $A, B \subseteq V(\vec{G})$, we write $\vec{G}[A, B]$ for the bipartite subdigraph of \vec{G} with vertex classes A and B and whose edges are all the edges of \vec{G} whose initial vertex belongs to A and whose final vertex belongs to B .

The *length* of a path is number of edges it contains. An (x, y) -*path* is a path whose endpoints are x and y . Given a path P and $x, y \in V(P)$, we write xPy for the (x, y) -path induced by P . We use the terms *set of vertex-disjoint paths* and *linear forest* interchangeably. In particular, by slightly abusing notation, given a set \mathcal{P} of vertex-disjoint paths, we write $V(\mathcal{P})$ for the set of vertices of the paths in \mathcal{P} and define $E(\mathcal{P})$ similarly.

We write \mathbb{N} for the set of natural numbers (including 0) and \mathbb{N}^* for the set of positive natural numbers. For any $k \in \mathbb{N}^*$, we write $[k] := \{1, 2, \dots, k\}$, $[k]_{\text{odd}} := \{i \in [k] \mid k \text{ is odd}\}$, and, similarly, $[k]_{\text{even}} := \{i \in [k] \mid k \text{ is even}\}$.

Let $a, b, c \in \mathbb{R}$. We write $a = b \pm c$ if $b - c \leq a \leq b + c$. For simplicity, we use hierarchies instead of explicitly calculating the values of constants for which statements hold. Namely, if we write $0 < a \ll b \ll c \leq 1$ in a statement, we mean that there exist non-decreasing functions $f: (0, 1] \rightarrow (0, 1]$ and $g: (0, 1] \rightarrow (0, 1]$ such that the statement holds for all $0 < a, b, c \leq 1$ satisfying $b \leq f(c)$ and $a \leq g(b)$. Hierarchies with more constants are defined in a similar way. We assume large numbers to be integers and omit floors and ceilings, provided this does not affect the argument.

Let G be a graph. A *decomposition* of G is a set \mathcal{D} of edge-disjoint subgraphs of G such each edge of G belongs to exactly one subgraph in \mathcal{D} . A *path decomposition* (respectively, *cycle decomposition*) is a decomposition \mathcal{D} of G such that each subgraph in \mathcal{D} is a path (respectively,

a cycle). We say G can be *decomposed into d paths* (respectively, *decomposed into d cycles*) if G has a path (respectively, cycle) decomposition \mathcal{D} of size d . Similarly, we say G can be *decomposed into d paths and cycles* (respectively, *decomposed into d cycles and edges*) if G has a decomposition \mathcal{D} of size d such that each subgraph in \mathcal{D} is either a path or a cycle (a cycle or an edge, respectively).

3.2. Regularity. Let G be a bipartite graph on vertex classes A, B . The *density* of G is $d_G(A, B) := \frac{e_G(A, B)}{|A||B|}$. We may write $d(A, B)$ instead of $d_G(A, B)$ if this is unambiguous. For any $\varepsilon > 0$, we say G is ε -*regular* if, for any $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$, we have $|d(A', B') - d(A, B)| < \varepsilon$.

Let $d \in [0, 1]$. We say G is (ε, d) -*regular* if G is ε -regular and has density d . We write G is $(\varepsilon, \geq d)$ -*regular* is G is ε -regular of density at least d . We say G is $[\varepsilon, d]$ -*superregular* if G is ε -regular, for all $a \in A$, $d(a) = (d \pm \varepsilon)|B|$, and, for all $b \in B$, $d(b) = (d \pm \varepsilon)|A|$. We say G is $[\varepsilon, \geq d]$ -*superregular* if there exists $d' \geq d$ such that G is $[\varepsilon, d']$ -superregular.

We also define a sparse version of ε -(super)regularity to allow for $d < \varepsilon$. Let G be a bipartite graph on vertex classes A, B of size m . We say G is $\{\varepsilon, d\}$ -*regular* if for any $A' \subseteq A$ and $B' \subseteq B$ with $|A'|, |B'| \geq \varepsilon m$, we have $d(A', B') = (1 \pm \varepsilon)d$. For any $0 < c < 1$, we say G is (ε, d, c) -*regular* if the following hold:

- (Reg 1) G is $\{\varepsilon, d\}$ -regular;
- (Reg 2) For any distinct $a, a' \in A$ we have $|N(a) \cap N(a')| \leq c^2 m$, and similarly $|N(b) \cap N(b')| \leq c^2 m$ for any distinct $b, b' \in B$;
- (Reg 3) $\Delta(G) \leq cm$.

For any $0 < d^* < 1$, we say that G is (ε, d, d^*, c) -*superregular* if it is (ε, d, c) -regular and the following holds:

- (Reg 4) $\delta(G) \geq d^* m$.

Given a bipartite digraph \vec{G} with vertex classes A, B , recall that $\vec{G}[A, B]$ denotes the bipartite subgraph of \vec{G} whose edges are all the edges directed from A to B in \vec{G} . We often view $\vec{G}[A, B]$ as an undirected bipartite graph. In particular, we say $\vec{G}[A, B]$ is ε -regular if this holds when $\vec{G}[A, B]$ is viewed as an undirected graph. We define (ε, d) -regularity, $(\varepsilon, \geq d)$ -regularity, $[\varepsilon, d]$ -superregularity, and $[\varepsilon, \geq d]$ -superregularity for directed bipartite graphs similarly.

Let G be a graph and V_0, V_1, \dots, V_k be a partition of $V(G)$ into k clusters V_1, \dots, V_k and an *exceptional set* V_0 . The vertices in V_0 are called the *exceptional vertices* of G and an edge of G is called *exceptional* if it has an endpoint in V_0 . The *reduced graph* of G (with respect to the partition V_0, V_1, \dots, V_k) is the graph R with $V(R) := \{V_1, \dots, V_k\}$ and $E(R) := \{V_i V_j \mid e(G[V_i, V_j]) > 0\}$. For clarity, we sometimes abuse notation and denote by $1, \dots, k$ the vertices of R . If C is a connected component of R , we let $V_G(C) := \bigcup C$, i.e. $V_G(C)$ is the set of vertices $x \in V(G)$ such that $x \in V_i$ for some $V_i \in C$. The *reduced digraph* \vec{R} of a digraph \vec{G} is defined similarly.

Let G be an n -vertex graph. Let V_0, V_1, \dots, V_k be a partition of $V(G)$ and R be the corresponding reduced graph. For any distinct $i, j \in [k]$, the *support cluster of V_i with respect to V_j* is the set $V_{ij} := \{x \in V_i \mid N_G(x) \cap V_j \neq \emptyset\}$. We also say V_{ij} and V_{ji} are the *support clusters of the pair $G[V_i, V_j]$* . Let $ij \in E(R)$ and $x \in V_i$. We say x *belongs to the superregular pair $G[V_i, V_j]$* if x belongs to the support cluster V_{ij} . Let $V'_0, V'_1, \dots, V'_{k'}$ be a partition of $V(G)$ such that, for all $i \in [k']$, there exists $j \in [k]$ such that $V'_i \subseteq V_j$. We say the support clusters of the partition $V'_0, V'_1, \dots, V'_{k'}$ are *induced by* the partition V_0, V_1, \dots, V_k if, for all $i', j' \in [k']$, the support cluster $V'_{i'j'}$ of $V'_{i'}$ with respect to $V'_{j'}$ satisfies $V'_{i'j'} = V_{ij} \cap V'_{i'}$, where $i, j \in [k]$ are such that $V'_{i'} \subseteq V_i$, $V'_{j'} \subseteq V_j$, and $V_{ij} := \emptyset$ if $i = j$. Let G' be a graph on $V(G)$ with reduced graph R' (with respect to the partition V_0, V_1, \dots, V_k). We say G and G' *have the same support clusters* if for any $ij \in E(R) \cap E(R')$, the support clusters of the pairs $G[V_i, V_j]$ and $G'[V_i, V_j]$ are the same.

We say V_0, V_1, \dots, V_k is an $(\varepsilon, \geq d, k, m, R)$ -*superregular partition of G* if the following hold.

- (SRP1) $|V_1| = \dots = |V_k| = m$.
- (SRP2) $|V_0| \leq \varepsilon n$.
- (SRP3) $G[V_i]$ is empty for all $i \in [k]$.
- (SRP4) R is the reduced graph of G .
- (SRP5) For any $ij \in E(R)$, let V_{ij}, V_{ji} be the support clusters of $G[V_i, V_j]$. Then, $G[V_{ij}, V_{ji}]$ is $[\varepsilon, \geq d]$ -superregular and $|V_{ij}|, |V_{ji}| \geq (1 - \varepsilon)m$.

We say V_0, V_1, \dots, V_k is an $(\varepsilon, \geq d, k, m, m', R)$ -superregular equalised partition of G if (SRP1)–(SRP5) are satisfied and, moreover, the following holds.

- (SRP6) $m' \geq (1 - \varepsilon)m$ and, for any $ij \in E(R)$, $|V_{ij}| = |V_{ji}| = m'$.

We say V_0, V_1, \dots, V_k is an $(\varepsilon, d, k, m, R)$ -superregular partition of G if (SRP1)–(SRP5) hold, except that $[\varepsilon, \geq d]$ -superregularity is replaced by $[\varepsilon, d]$ -superregularity in (SRP5). We define an $(\varepsilon, d, k, m, m', R)$ -superregular equalised partition of G analogously.

We say a graph G admits a superregular (equalised) partition if there exist V_0, V_1, \dots, V_k , ε, d, k, m, R (and m') such that V_0, V_1, \dots, V_k is an $(\varepsilon, \geq d, k, m, R)$ -superregular (equalised) partition.

3.3. Probabilistic estimates. Let X be a random variable. We write $X \sim \text{Bin}(n, p)$ if X follows a binomial distribution with parameters n, p . Let $N, n, m \in \mathbb{N}$ be such that $\max\{n, m\} \leq N$. Let Γ be a set of size N and $\Gamma' \subseteq \Gamma$ be of size m . Recall that X has a *hypergeometric distribution with parameters N, n, m* if $X = |\Gamma_n \cap \Gamma'|$, where Γ_n is a random subset of Γ with $|\Gamma_n| = n$ (i.e. Γ_n is obtained by drawing n elements of Γ without replacement). We will denote this by $X \sim \text{Hyp}(N, n, m)$.

We will use the following Chernoff-type bound.

Lemma 3.1 (see e.g. [23, Theorem 2.1 and Theorem 2.10]). *Assume $X \sim \text{Bin}(n, p)$ or $X \sim \text{Hyp}(N, n, m)$. Then the following hold for any $0 < \varepsilon < 1$:*

- (i) $\mathbb{P}[X \leq (1 - \varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$;
- (ii) $\mathbb{P}[X \geq (1 + \varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$.

4. PRELIMINARY RESULTS

In this section, we introduce some preliminary results which will be useful in the proof of our main theorems. In Sections 4.1 and 4.2, we collect some useful properties of ε -regular pairs and prove some lemmas for tying paths together. These results will be used repeatedly in the rest of the paper. In Sections 4.3 and 4.4, we introduce some tools for regularising superregular pairs and state the robust decomposition lemma of [27], which will be needed in Section 5.5.

4.1. Regularity. The first lemma follows easily from the definition of ε -regularity.

Lemma 4.1. *Let $0 < \frac{1}{m} \ll \varepsilon \leq d < 1$ and assume $\varepsilon \leq \eta \leq 1$. Let G be a (ε, d) -regular bipartite graph on vertex classes A, B of size m . If $A' \subseteq A$, $B' \subseteq B$ have size at least ηm , then $G[A', B']$ is $\frac{\varepsilon}{\eta}$ -regular of density $\geq d - \varepsilon$.*

The following lemma states that ε -regularity is preserved if only few vertices and edges are removed. This result will be used repeatedly in the rest of the paper.

Lemma 4.2 ([27, Proposition 4.3]). *Let $0 < \frac{1}{m} \ll \varepsilon \leq d' \leq d \leq 1$ and G be a bipartite graph on vertex classes of size m . Suppose G' is obtained from G by removing at most $d'm$ vertices from each vertex class and at most $d'm$ edges incident to each vertex from G .*

- (i) *If G is (ε, d) -regular, then G' is $(2\sqrt{d'}, \geq d - 2\sqrt{d'})$ -regular.*
- (ii) *If G is $[\varepsilon, d]$ -superregular, then G' is $[2\sqrt{d'}, d]$ -superregular.*

An analogous result holds for the sparse version of regularity.

Lemma 4.3 ([27, Proposition 4.8]). *Let $0 < \frac{1}{m} \ll d' \ll \varepsilon, d, d^*, c \leq 1$. Let G be an (ε, d, d^*, c) -superregular bipartite graph on vertex classes of size m . Suppose G' is obtained from G by removing at most $d'm$ edges incident to each vertex from G . Then G' is $(2\varepsilon, d, d^* - d', c)$ -superregular.*

The following proposition states an ε -regular bipartite graph of linear minimum degree has small diameter.

Proposition 4.4. *Let $0 < \frac{1}{m_A}, \frac{1}{m_B} \ll \varepsilon \leq d \leq 1$. Let G be an ε -regular bipartite graph on vertex classes A and B of size m_A and m_B . Suppose that each $x \in A$ satisfies $d_G(x) \geq dm_B$ and each $y \in B$ satisfies $d_G(y) \geq dm_A$. Then, for any $x \in A$ and $y \in B$, G contains an (x, y) -path of length at most 3. In particular, for any distinct $x, y \in V(G)$, G contains an (x, y) -path of length at most 4.*

The following lemma states that balanced ε -regular bipartite graphs of large minimum degree are Hamiltonian.

Lemma 4.5 (see for instance [20, Lemma 3.3]). *Let $0 < \frac{1}{m} \ll \varepsilon \ll \alpha \leq 1$. If G is an ε -regular bipartite graph on vertex classes of size m such that $\delta(G) \geq \alpha m$, then G contains a Hamilton cycle.*

Corollary 4.6. *Let $0 < \frac{1}{m} \ll \varepsilon \ll \alpha \leq 1$. If G is an ε -regular bipartite graph on vertex classes of size m such that $\delta(G) \geq \alpha m$, then G contains a perfect matching.*

The next lemma states that any superregular pair contains a sparse superregular pair as a subgraph.

Lemma 4.7 ([27, Lemma 4.10]). *Let $0 < \frac{1}{m} \ll \varepsilon, d' \leq d \leq 1$ and suppose $\varepsilon \ll d$. Let G be an $[\varepsilon, d]$ -superregular bipartite graph on vertex classes of size m . Then G contains an $(\varepsilon^{\frac{1}{12}}, d', \frac{d'}{2}, \frac{3d'}{2d})$ -superregular spanning subgraph.*

By considering a random partition of the edges, one can show that the edges of an ε -regular pair can be partitioned without destroying the ε -regularity (see e.g. the proof of [27, Lemma 4.10]).

Lemma 4.8 (Partitioning the edges of a regular pair). *Assume $0 < \frac{1}{m} \ll \varepsilon \ll d_1, \dots, d_\ell \leq d \leq 1$ with $\sum_{i=1}^\ell d_i \leq d$. Let G be a bipartite graph on vertex classes A, B of size m . Then G can be decomposed into edge-disjoint spanning subgraphs $G_0, G_1, \dots, G_\ell \subseteq G$ such that G_0 is empty if $\sum_{i=1}^\ell d_i = d$, and the following hold for each $i \in [\ell]$.*

- (i) *If G is (ε, d) -regular, then G_i is $(\varepsilon^{\frac{1}{12}}, d_i \pm \varepsilon^{\frac{1}{12}})$ -regular.*
- (ii) *If G is $[\varepsilon, d]$ -superregular, then G_i is $[\varepsilon^{\frac{1}{12}}, d_i]$ -superregular.*

Corollary 4.9. *Suppose $0 < \frac{1}{m} \ll \varepsilon \ll d \leq 1$. Let G be an $[\varepsilon, d]$ -superregular bipartite graph on vertex classes A and B of size m . Then, there exists an orientation \vec{G} of the edges of G such that both $\vec{G}[A, B]$ and $\vec{G}[B, A]$ are $[\varepsilon^{\frac{1}{12}}, \frac{d}{2}]$ -superregular.*

Using Lemma 3.1 and a result of [1] which characterises ε -regularity in terms of co-degree, one can show that the vertex classes of superregular pairs can be partitioned into superregular subpairs.

Lemma 4.10 (Partitioning the vertices of a regular pair). *Assume $0 < \frac{1}{m} \ll \varepsilon \ll d \leq 1$ and $\frac{1}{m} \ll \frac{1}{r}, \eta$. Let G be a bipartite graph on vertex classes A and B of size m . Let A' and B' be the support clusters of A and B , respectively. Assume that $m' := |A'| = |B'| \geq (1 - \varepsilon)m$ and $G[A', B']$ is $[\varepsilon, d]$ -superregular. Let $m_1, \dots, m_r \in \mathbb{N}^*$ be such that $\sum_{i \in [r]} m_i = m$ and, for each $i \in [r]$, $m_i = \frac{m}{r} \pm 1$. Assume A and B are randomly partitioned into r subsets A_1, \dots, A_r and B_1, \dots, B_r such that for all $i \in [r]$, $|A_i| = |B_i| = m_i$. Then, with high probability, all of the following hold.*

- (i) *For any $i \in [r]$, we have $|A' \cap A_i| = (1 \pm \varepsilon)\frac{m'}{r}$ and similarly $|B' \cap B_i| = (1 \pm \varepsilon)\frac{m'}{r}$.*

- (ii) $G[X, Y]$ is $[\varepsilon^{\frac{1}{7}}, d]$ -superregular for any $X \in \{A' \cap A_i, A' \setminus A_i \mid i \in [r]\}$ and $Y \in \{B' \cap B_i, B' \setminus B_i \mid i \in [r]\}$.
- (iii) For any $X \in \{A' \cap A_i \mid i \in [r]\}$ and $Y \in \{B' \cap B_i \mid i \in [r]\}$, $\Delta(G[X, Y]) \leq \frac{\Delta(G) + \eta m'}{r}$ and $\delta(G[X, Y]) \geq \frac{\delta(G) - \eta m'}{r}$.

Finally, the following simple fact will be needed in Section 5.5.

Proposition 4.11. *Suppose $0 \leq \frac{1}{m} \ll \varepsilon \ll d \leq 1$ and $k \in \mathbb{N}^*$. Let G be a graph and V_1, \dots, V_k be a partition of $V(G)$ into k clusters of size m . Let R be the corresponding reduced graph of G and assume that for each $ij \in E(R)$, the pair $G[V_i, V_j]$ is $[\varepsilon, \geq d]$ -superregular. If R is a cycle of length k , then G contains εm vertex-disjoint cycles of length k which intersect each of the clusters V_1, \dots, V_k .*

4.2. Tying paths together. Throughout the proof of our main theorems, we will form linear forests and aim to tie together some of the paths in each forest to form cycles. This section gathers several tools to achieve this. Lemma 4.12 will be used to efficiently reduce the number of components of linear forests (i.e. to merge paths), from a linear number of components to bounded number, while Lemma 4.13 will be used to further reduce the number of components, from a large constant to a smaller one. Lemma 4.16 will be used to turn linear forests with few components into small sets of vertex-disjoint cycles. Finally, we will use Lemmas 4.14, 4.15, and 4.19 to turn small linear forests into a cycle each.

Let Γ be a graph and P_1, \dots, P_ℓ be vertex-disjoint paths with endpoints in $V(\Gamma)$. By *tying the paths P_1, \dots, P_ℓ together into a path P (a cycle C) using the edges of Γ* , we mean forming a path P (a cycle C) such that for each $i \in [\ell]$, the path P_i is a subpath of P (of C), the other edges of P (of C) are edges of Γ and the endpoints of P are in $\bigcup_{i \in [\ell]} V(P_i)$. A subpath P' of P (of C) is called a *link path* if $E(P') \cap E(P_i) = \emptyset$ for each $i \in [\ell]$ and the endpoints of P' are in $\bigcup_{i \in [\ell]} V(P_i)$. In particular, we say P' *links* P_i and P_j if the endpoints of P' are an endpoint of P_i and an endpoint of P_j . Moreover, if A, B are distinct clusters, we say P' is an (A, B) -*link path* if $E(P') \subseteq E(\Gamma[A, B])$ and both endpoints of P' belong to A .

The idea behind the next lemma is to iteratively tie two paths which have an endpoint in a common cluster using a single superregular pair of Γ .

Lemma 4.12. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \varepsilon \leq \zeta \ll \beta \leq 1$. Let Γ be a graph on vertex set V of size n such that the following hold.*

- (i) V_0, V_1, \dots, V_k is an $(\varepsilon, \beta, k, m, R)$ -superregular partition of Γ .
- (ii) Any $x \in V \setminus V_0$ belongs to at least βk superregular pairs of Γ .

Let $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ be sets of paths satisfying the following.

- (iii) For each $i \in [\ell]$, \mathcal{P}_i is a set of vertex-disjoint paths with endpoints in $V \setminus V_0$.
- (iv) For each $i \in [\ell]$ and $j \in [k]$, $|V(\mathcal{P}_i) \cap V_j| \leq \zeta m$. In particular, $|\mathcal{P}_i| \leq \zeta n$.
- (v) For any $x \in V$, there are at most εn paths in $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell$ which have x as an endpoint.

Then, there exist disjoint $E_1, \dots, E_\ell \subseteq E(\Gamma)$ such that the following hold.

- (a) For any $i \in [\ell]$, by using each edge in E_i exactly once, we can tie together some of the paths in \mathcal{P}_i to form a set \mathcal{Q}_i of vertex-disjoint paths such that, for any $j \in [k]$, at most $2\beta^{-2}$ paths in \mathcal{Q}_i have an endpoint in V_j .
- (b) For any distinct $i, j \in [k]$ and $x \in V_i$, $E_1 \cup \dots \cup E_\ell$ contains at most $3\varepsilon^{\frac{1}{4}}m$ edges of $\Gamma[V_i, V_j]$ which are incident to x .
- (c) For any $i \in [\ell]$ and $j \in [k]$, $|V(\mathcal{P}_i \cup E_i) \cap V_j| \leq \sqrt{\zeta}m$.

To prove Lemma 4.12, we will use edges of Γ to tie together some of the paths in \mathcal{P}_i , for each $i \in [\ell]$. We will only tie together paths which have an endpoint in a common cluster and use a single superregular pair of Γ to do so.

Proof. Let $E_1, \dots, E_\ell \subseteq E(\Gamma)$ be (possibly empty) disjoint sets of edges of Γ and assume inductively that for each $i \in [\ell]$, by using each edge in E_i exactly once, we can tie together some of the paths in \mathcal{P}_i to form a set \mathcal{Q}_i of vertex-disjoint paths such that the following is satisfied.

- (1) If $P \in \mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ and P' is a link path of P , then P' is an (A, B) -link path of length at most 4, for some clusters A, B .
- (2) For any clusters A, B and any $x \in A$, there are at most $\varepsilon^{\frac{1}{2}}m$ (A, B) -link paths in $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ which have x as an endpoint.
- (3) For any clusters A, B and any $x \in A \cup B$, $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ contains at most $\varepsilon^{\frac{1}{4}}m$ (A, B) -link paths which have x as an internal vertex.
- (4) For any cluster A and $i \in [\ell]$, there are at most $\frac{\sqrt{\zeta}m}{4}$ tuples (B, P) such that $B \neq A$ is a cluster and P is a (B, A) -link path in \mathcal{Q}_i .

(In (2)–(4) and below, by a link path in \mathcal{Q}_i , we mean a link path of some path in \mathcal{Q}_i .)

If for any $i \in [\ell]$ and $j \in [k]$, the set \mathcal{Q}_i contains at most $2\beta^{-2}$ paths with an endpoint in V_j , then (a) holds. Moreover, (2) and (3) imply (b), while (c) follows from (1), (4), and (iv), and we are done.

We may therefore assume that there exist $i \in [\ell]$ and $j \in [k]$ such that \mathcal{Q}_i contains more than $2\beta^{-2}$ paths with an endpoint in V_j . Then, we claim that there exist distinct $P, P' \in \mathcal{Q}_i$, each with an endpoint in V_j , such that the following hold. There exists $j' \in [k]$ such that

- (I) $x \in V_j$ is an endpoint of P and $x' \in V_j$ is an endpoint of P' ;
- (II) $x, x' \in V_{jj'}$, where $V_{jj'}$ is the support cluster of V_j with respect to $V_{j'}$;
- (III) $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ contains fewer than $\varepsilon^{\frac{1}{2}}m$ $(V_j, V_{j'})$ -link paths which have x as an endpoint, and similarly for x' ;
- (IV) there are fewer than $\frac{\sqrt{\zeta}m}{4}$ tuples (A, Q) such that $A \neq V_{j'}$ is a cluster and Q is an $(A, V_{j'})$ -link path in \mathcal{Q}_i .

Indeed, for any $x \in V_j$, there are at least βk indices $j' \in [k]$ such that $x \in V_{jj'}$ (by (ii)). By (v), there are at most $2\varepsilon^{\frac{1}{2}}k$ such indices $j' \in [k]$ such that $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ contains $\varepsilon^{\frac{1}{2}}m$ $(V_j, V_{j'})$ -link paths with x as an endpoint. Moreover, by (iv), there are at most $5\sqrt{\zeta}k$ indices $j' \in [k]$ such that there exist $\frac{\sqrt{\zeta}m}{4}$ tuples (A, Q) where $A \neq V_{j'}$ is a cluster and Q is an $(A, V_{j'})$ -link path in \mathcal{Q}_i . Thus, for any $x \in V_j$, there are at least $\frac{\beta^2 k}{2}$ indices j' such that x satisfies (II)–(IV). Therefore, since by assumption \mathcal{Q}_i contains more than $2\beta^{-2}$ paths with an endpoint in V_j , we can find P, P', x, x' , and j' satisfying (I)–(IV).

We can now find an (x, x') -path in $\Gamma[V_{jj'}, V_{j'j}]$ to tie P and P' together as follows. Let $\Gamma' := \Gamma \setminus (E_1 \cup \dots \cup E_\ell)$. By (2) and (3), Lemma 4.2 implies that $\Gamma'[V_{jj'}, V_{j'j}]$ is still $[\varepsilon^{\frac{1}{9}}, \beta]$ -superregular. Let V'_j be obtained from $V_{jj'}$ by deleting the following vertices:

- vertices in $V(\mathcal{Q}_i) \setminus \{x, x'\}$ (by (1), (iv), and (4), there are at most $\sqrt{\zeta}m$ such vertices);
- vertices in $V_{jj'} \setminus \{x, x'\}$ which are an internal vertex of $\varepsilon^{\frac{1}{4}}m$ $(V_j, V_{j'})$ -link paths of $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ (by (1) and (2), there are at most $\varepsilon^{\frac{1}{4}}m$ such vertices).

Note that the number of deleted vertices is $|V_{jj'} \setminus V'_j| \leq \sqrt{\zeta}m + \varepsilon^{\frac{1}{4}}m \leq 2\sqrt{\zeta}|V_{jj'}|$. Define $V'_{j'}$ similarly. Then, by Lemma 4.2, $\Gamma'[V'_j, V'_{j'}]$ is $[\zeta^{\frac{1}{5}}, \beta]$ -superregular. Thus, by Proposition 4.4, $\Gamma'[V'_j, V'_{j'}]$ contains an (x, x') -path P'' of length at most 4. Add the edges of P'' to E_i and replace in \mathcal{Q}_i the paths P and P' by the concatenation of P, P'' , and P' . By construction, (1)–(4) are still satisfied, as desired for the induction step. \square

After applying Lemma 4.12, we obtain linear forests with few components. One can then be less economical and use several superregular pairs of Γ to tie paths together. This is achieved in the next lemma.

Lemma 4.13. Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \varepsilon \leq \zeta \ll \beta \leq 1$. Let Γ be a graph on vertex set V of size n and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ be sets of paths. Assume Γ and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ satisfy properties (i)–(v) of Lemma 4.12, as well as the following.

- (vi) $\ell \leq n$.
- (vii) For any $i \in [\ell]$ and $j \in [k]$, \mathcal{P}_i contains at most $2\beta^{-2}$ paths with an endpoint in V_j (and thus at most $2\beta^{-2}k$ paths in total).

Then, there exist disjoint $E_1, \dots, E_\ell \subseteq E(\Gamma)$ such that the following hold.

- (a) For any $i \in [\ell]$, by using each edge in E_i exactly once, we can tie together some of the paths in \mathcal{P}_i to form a set \mathcal{Q}_i of vertex-disjoint paths such that for any connected component C of R , \mathcal{Q}_i contains at most one path with an endpoint in $V_\Gamma(C)$.
- (b) For any distinct $i, j \in [k]$ and $x \in V_i$, $E_1 \cup \dots \cup E_\ell$ contains at most $3\varepsilon^{\frac{1}{4}}m$ edges of $\Gamma[V_i, V_j]$ which are incident to x .
- (c) For any $i \in [\ell]$ and $j \in [k]$, $|V(\mathcal{P}_i \cup E_i) \cap V_j| \leq \sqrt{\zeta}m$.

This is proved similarly to Lemma 4.12 but since we now have fewer paths to link, we can use several superregular pairs of Γ to tie together paths whose endpoints are not necessarily in a same cluster. Thus, the main difference to the proof of Lemma 4.12 is that, in order to link two paths, we no longer need to find a suitable superregular pair of Γ but a suitable walk in the reduced graph of Γ . Moreover, since we have few paths to tie together, we no longer need to ensure that no superregular pair is used too many times (condition (4) in the proof of Lemma 4.12). Finally, note that since link paths may now intersect several superregular pairs of Γ , it no longer makes sense to talk about (A, B) -link paths, so we only use the generic term *link path* (defined at the beginning of Section 4.2).

Proof. Let $E_1, \dots, E_\ell \subseteq E(\Gamma)$ be (possibly empty) disjoint sets of edges of Γ and assume inductively that for each $i \in [\ell]$, by using each edge in E_i exactly once, we can tie together some of the paths in \mathcal{P}_i to form a set \mathcal{Q}_i of vertex-disjoint paths such that the following is satisfied.

- (1) If $P \in \mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ and P' is a link path of P , then P' contains at most 3 vertices from each cluster and at most 4 edges from each superregular pair of Γ .
- (2) For any clusters A and B , and any $x \in A$, the set $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ contains at most $\varepsilon^{\frac{1}{2}}m$ link paths which have x as an endpoint and whose edge incident to x belongs to $\Gamma[A, B]$.
- (3) For any $x \in V(\Gamma)$, there are at most $\varepsilon^{\frac{1}{4}}m$ link paths in $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ which contain x as an internal vertex.

If for any $i \in [\ell]$ and any connected component C of R , the set \mathcal{Q}_i contains at most one path with an endpoint in $V_\Gamma(C)$, then (a) holds. Moreover, (2) and (3) imply (b), while (c) follows from (1), (iv), and (vii), and we are done.

We may therefore assume that there exist $i \in [\ell]$, a component C of R , distinct paths $P, P' \in \mathcal{Q}_i$ and distinct vertices $x, x' \in V_\Gamma(C)$ such that x and x' are endpoints of P and P' , respectively. We find an (x, x') -path in Γ to link P and P' as follows. Let $\Gamma' := \Gamma \setminus (E_1 \cup \dots \cup E_\ell)$. By (2) and (3), Lemma 4.2 implies that for any $jj' \in E(R)$, $\Gamma'[V_{jj'}, V_{j'j}]$ is still $[\varepsilon^{\frac{1}{9}}, \beta]$ -superregular, where $V_{jj'}$ and $V_{j'j}$ are the support clusters of $\Gamma[V_j, V_{j'}]$.

Let $i', i'' \in [k]$ be such that $x \in V_{i'}$ and $x' \in V_{i''}$. Choose $j' \in [k]$ such that $x \in V_{i'j'}$ and $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ contains fewer than $\varepsilon^{\frac{1}{2}}m$ link paths which have x as an endpoint and whose edge incident to x belongs to $\Gamma[V_{i'}, V_{j'}]$. The existence of such an index j' is guaranteed by (ii) and (v). Indeed, by (v), there are at most $2\varepsilon^{\frac{1}{2}}k < \beta k$ indices j' such that $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ contains $\varepsilon^{\frac{1}{2}}m$ link paths which have x as an endpoint and whose edge incident to x belongs to $\Gamma[V_{i'}, V_{j'}]$. The existence of the desired index j' now follows from (ii). Similarly, pick $j'' \in [k]$ such that $x' \in V_{i''j''}$ and $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ contains fewer than $\varepsilon^{\frac{1}{2}}m$ link paths which have x' as an endpoint and whose edge incident to x' belongs to $\Gamma[V_{i''}, V_{j''}]$.

Let $V_{i_1} \dots V_{i_r}$ be a $(V_{j'}, V_{j''})$ -path in R , where $i_1 := j'$ and $i_r := j''$. Let $i_0 := i'$ and $i_{r+1} := i''$. Then, $V_{i_0} V_{i_1} \dots V_{i_r} V_{i_{r+1}}$ is a $(V_{i'}, V_{i''})$ -walk in R where the clusters $V_{i'}$ and $V_{i''}$ appear at most twice and all other clusters occur at most once. For $0 \leq s \leq r+1$, let V'_{i_s} be obtained from V_{i_s} by deleting the following vertices:

- vertices in $V_{i_s} \setminus (V_{i_{s-1}i_s} \cap V_{i_si_{s+1}})$ (by (i), there are at most $2\epsilon m$ such vertices);
- vertices in $V(\mathcal{Q}_i) \setminus \{x, x'\}$ (by (1), (iv), and (vii), there are at most $\frac{3\zeta m}{2}$ such vertices);
- vertices in $V_{i_s} \setminus \{x, x'\}$ which are an internal vertex of $\epsilon^{\frac{1}{4}}m$ link paths in $\mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_\ell$ (by (1) and (vi), there are at most ϵm such vertices).

Then, $|V'_{i_s}| \geq m - 2\zeta m$. So for any $s \in [r+1]$, by Lemma 4.2, $\Gamma'[V'_{i_{s-1}}, V'_{i_s}]$ is still $[\zeta^{\frac{1}{3}}, \beta]$ -superregular. We can therefore find an (x, x') -path P'' in Γ' containing exactly one edge of $\Gamma'[V'_{i_{s-1}}, V'_{i_s}]$ for each $s \in [r]$ and at most 3 edges of $\Gamma'[V'_{i_r}, V'_{i_{r+1}}]$. We add the edges of P'' to E_i and replace in \mathcal{Q}_i the paths P, P' by the concatenation of P, P'' , and P' . By construction, (1)–(3) are still satisfied, as desired. \square

The methods used to prove the previous lemma can be used to close a path P into a cycle provided the endpoints of P lie in a same connected component of Γ . More generally, one can show the following.

Lemma 4.14. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \epsilon \leq \zeta \ll \beta \leq 1$. Let Γ be a graph on vertex set V of size n and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ be sets of paths. Assume Γ and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ satisfy properties (i)–(vii) of Lemmas 4.12 and 4.13. Suppose moreover that the following holds.*

- (viii) *For each $i \in [\ell]$, there exists an ordering $P_{i,1}, \dots, P_{i,\ell_i}$ of the paths in \mathcal{P}_i , and, for each $j \in [\ell_i]$, an ordering $x_{i,j}, x'_{i,j}$ of the endpoints of $P_{i,j}$ such that the following holds. For each $i \in [\ell]$ and $j \in [\ell_i]$, there exists a component C of R such that $x'_{i,j}, x_{i,j+1} \in V_\Gamma(C)$, where $x_{i,\ell_i+1} := x_{i,1}$.*

Then, there exist disjoint $E_1, \dots, E_\ell \subseteq E(\Gamma)$ such that the following hold.

- (a) *For any $i \in [\ell]$, $\mathcal{P}_i \cup E_i$ forms a cycle.*
- (b) *For any distinct $i, j \in [k]$ and $x \in V_i$, $E_1 \cup \dots \cup E_\ell$ contains at most $3\epsilon^{\frac{1}{4}}m$ edges of $\Gamma[V_i, V_j]$ which are incident to x .*

Proof. The idea is to link the paths $P_{i,j}$ and $P_{i,j+1}$ together for each $i \in [\ell]$ and $j \in [\ell_i]$, where $P_{i,\ell_i+1} := P_{i,1}$. This can be done by using the arguments of Lemma 4.13 to find an $(x'_{i,j}, x_{i,j+1})$ -path in Γ for each $i \in [\ell]$ and $j \in [\ell_i]$. \square

In general, our sets of paths will not satisfy property (viii) of Lemma 4.14. In that case, we need to add suitable edges to our sets of paths before applying Lemma 4.14. This is achieved in the next lemma.

Lemma 4.15. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \epsilon \ll \zeta \ll \beta \leq 1$. Let Γ be a graph on vertex set V of size n and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ be sets of paths. Assume Γ and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ satisfy properties (i)–(vi) of Lemmas 4.12 and 4.13, as well as the following.*

- (vii') *For any $i \in [\ell]$ and any connected component C of R , \mathcal{P}_i contains at most one path with an endpoint in $V_\Gamma(C)$.*

Let Γ' be a graph on V such that Γ and Γ' are edge-disjoint and the following hold.

- (ix) *V_0, V_1, \dots, V_k is an $(\epsilon, \zeta, k, m, R')$ -superregular partition of Γ' .*
- (x) *$R \cup R'$ is connected.*

Then, there exist disjoint $E_1, \dots, E_\ell \subseteq E(\Gamma)$ and $E'_1, \dots, E'_\ell \subseteq E(\Gamma')$ such that the following hold.

- (a) *For any $i \in [\ell]$, $\mathcal{P}_i \cup E_i \cup E'_i$ forms a cycle.*

- (b) For any distinct $i, j \in [k]$ and $x \in V_i$, $E_1 \cup \dots \cup E_\ell$ contains at most $3\varepsilon^{\frac{1}{4}}m$ edges of $\Gamma[V_i, V_j]$ which are incident to x and $E'_1 \cup \dots \cup E'_\ell$ contains at most εm edges of $\Gamma'[V_i, V_j]$ which are incident to x .

Proof. We add some edges of Γ' to each \mathcal{P}_i in order to satisfy property (viii) of Lemma 4.14 as follows. For any $i \in [\ell]$, denote $\mathcal{P}_i := \{P_{i,1}, \dots, P_{i,\ell_i}\}$ and, for each $j \in [\ell_i]$, denote by $x_{i,j}, x'_{i,j}$ the endpoints of $P_{i,j}$. For each $i \in [\ell]$, indices ranging in $[\ell_i]$ are taken modulo ℓ_i , in particular $\ell_i + 1 := 1$.

Assume inductively that for some $0 \leq i \leq \ell$, $E'_1, \dots, E'_i \subseteq E(\Gamma')$ are disjoint and satisfy the following.

- (1) For each $j \in [i]$, the edges in E'_j are vertex-disjoint from each other and from paths in \mathcal{P}_j . In particular, $\mathcal{P}'_j := \mathcal{P}_j \cup E'_j$ is a set of vertex-disjoint paths.
- (2) For any distinct $j, j' \in [k]$ and $x \in V_j$, $E'_1 \cup \dots \cup E'_i$ contains at most εm edges of $\Gamma'[V_j, V_{j'}]$ which are incident to x .
- (3) For any $x \in V$, there are at most εn paths in $\mathcal{P}'_1 \cup \dots \cup \mathcal{P}'_i \cup \mathcal{P}_{i+1} \cup \dots \cup \mathcal{P}_\ell$ which have x as an endpoint.
- (4) For each $j \in [i]$, there exists a partition $E'_{j,1} \cup \dots \cup E'_{j,\ell_j}$ of E'_j such that the following holds. For each $j' \in [\ell_j]$, there exist an ordering $y_1 y'_1, \dots, y_t y'_t$ of the edges in $E'_{j,j'}$, and, distinct connected components C_0, \dots, C_t of R such that $x'_{j,j'} \in V_\Gamma(C_0)$, $x_{j,j'+1} \in V_\Gamma(C_t)$ and, for each $s \in [t]$, $y_s \in V_\Gamma(C_{s-1})$ and $y'_s \in V_\Gamma(C_s)$.

Assume $i = \ell$. Then, by (2), the second part of property (b) holds. Also note that $\mathcal{P}'_1, \dots, \mathcal{P}'_\ell$ satisfy conditions (i)–(viii) of Lemma 4.14, with 2ζ playing the role of ζ . Indeed, (i)–(iii) and (vi) are clearly satisfied. Moreover, by (ii), R has at most β^{-1} connected components and thus (4) and (vii') imply $|\mathcal{P}'_i| \leq (1 + \beta^{-1})|\mathcal{P}_i| \leq 2\beta^{-2}$. Therefore, (vii) holds. By (4), for each $i \in [\ell]$ and $j \in [k]$, we have $|V(\mathcal{P}'_i) \cap V_j| \leq |V(\mathcal{P}_i) \cap V_j| + 2|\mathcal{P}_i| \leq 2\zeta m$, so (iv) holds with 2ζ playing the role of ζ . Finally, (v) holds by (3) and (viii) follows from (4). Thus, we can apply Lemma 4.14 and we are done. We may therefore assume that $i < \ell$.

We construct E'_{i+1} as follows. Consider the auxiliary reduced graph \tilde{R} with the connected components of R as vertices and an edge between C and C' if R contains an edge between C and C' . Note that by (x), \tilde{R} is connected. For each $j \in [\ell_{i+1}]$, let C_j, C'_j be the connected components of R such that $x_{i+1,j} \in V_\Gamma(C_j)$ and $x'_{i+1,j} \in V_\Gamma(C'_j)$. For each $j \in [\ell_{i+1}]$, fix a (C'_j, C_{j+1}) -path Q_j in \tilde{R} .

Let Γ'' be obtained from Γ' by deleting the following edges:

- edges in $E'_1 \cup \dots \cup E'_i$ (by (2), we delete at most εm^2 such edges from each superregular pair of Γ');
- edges incident to some vertex x such that $\mathcal{P}'_1 \cup \dots \cup \mathcal{P}'_i \cup \mathcal{P}_{i+1} \cup \dots \cup \mathcal{P}_\ell$ contains εn paths which have x as an endpoint (by (vi) and since for all $i \in [\ell]$, $|\mathcal{P}_i| \leq |\mathcal{P}'_i| \leq 2\beta^{-2}$, we delete at most εm^2 such edges per superregular pair of Γ');
- edges incident to some vertex in $V(\mathcal{P}_{i+1})$ (by (iv) and (ix), we delete at most $3\zeta^2 m^2$ such edges from each superregular pair of Γ');
- for each $i', i'' \in [k]$, edges of $\Gamma'[V_{i'}, V_{i''}]$ which are incident to some vertex x such that $E'_1 \cup \dots \cup E'_i$ contains εm edges of $\Gamma'[V_{i'}, V_{i''}]$ incident to x (by the fact that R has at most β^{-1} connected components (by (ii)) as well as (4), (vi), (vii'), and (ix), we delete at most $2\varepsilon\zeta m^2$ such edges from each superregular pair of Γ').

Then, note that by (ix), for any $i' i'' \in E(R')$, $e(\Gamma''[V_{i'}, V_{i''}]) \geq (\zeta - \varepsilon)(1 - \varepsilon)^2 m^2 - \varepsilon m^2 - 3\zeta^2 m^2 - 2\varepsilon\zeta m^2 \geq \varepsilon m^2$. Thus, there exists a set $E'_{i+1} \subseteq E(\Gamma'')$ of vertex-disjoint edges of Γ'' such that (1)–(4) are still satisfied for $i = i + 1$, where, for $j = i + 1$ and each $j' \in [\ell_{i+1}]$, the components in $Q_{j'}$ play the roles of C_0, \dots, C_t in (4). \square

We will not always be able to add suitable edges to our sets of paths in order to apply Lemma 4.14. This problem can be circumvented by splitting paths and forming new sets of

paths. This is achieved in the next lemma. Note that the cost of this approach is that we may obtain more cycles than in Lemma 4.15, as well as a few leftover edges. Thus, Lemma 4.16 will only be used when we have some room to spare.

Lemma 4.16. *Suppose $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \leq \zeta \ll \beta \leq 1$. Let Γ be a graph on vertex set V of size n . Let $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ be sets of paths on V . Assume Γ and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ are all pairwise edge-disjoint and satisfy properties (i)–(iv) of Lemma 4.12, property (vii') of Lemma 4.15, as well as the following.*

(v') *For any $x \in V \setminus V_0$, $E(\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell)$ contains at most εn edges incident with x .*

(vi') $\ell \leq \zeta n$.

(xi) *For any $x \in V_0$, if xy and xy' are distinct edges in $E(\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell)$ then $y \in V_\Gamma(C)$ and $y' \in V_\Gamma(C')$ for some distinct components C and C' of R .*

Then, there exists $E \subseteq E(\Gamma)$ such that the following hold.

- (a) $(\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell) \cup E$ can be decomposed into a set \mathcal{C} of at most βn edge-disjoint cycles and a set E' of at most β^{-2} edges.
- (b) *For any distinct $i, j \in [k]$, and $x \in V_i$, E contains at most $\varepsilon^{\frac{1}{73}} m$ edges of $\Gamma[V_i, V_j]$ which are incident to x .*

To prove Lemma 4.16, we need the following results.

Theorem 4.17 (Vizing's theorem (see e.g. [6, Theorem 17.5])). *Let G be a multigraph with multiplicity $\mu(G)$. Then the edge-chromatic number $\chi'(G)$ of G satisfies $\chi'(G) \leq \Delta(G) + \mu(G)$. In particular, if G is simple, then $\chi'(G) \leq \Delta(G) + 1$.*

Lemma 4.18. *Assume G is a multigraph with maximum degree Δ , multiplicity μ , and $|E(G)|$ even. Then, G can be decomposed into at most $\frac{3(\Delta+\mu)}{2}$ matchings of even size and at most $\frac{\Delta+\mu}{2}$ paths and cycles of length 2.*

Proof. Let M_1, \dots, M_r be an optimal matching decomposition of G . By Theorem 4.17, $r \leq \Delta + \mu$. Let S be the set of indices $i \in [r]$ such that $|M_i| = 1$ and T be the set of indices $i \in [r]$ such that $|M_i|$ is odd and at least 3. If $|S|$ is odd, remove some $i \in S$ and add it to T so that $|S|$ is now even. Note that since $|E(G)|$ is even, $|T|$ must also be even.

For any distinct $i, j \in S$, by minimality of r , $M_i \cup M_j$ is either a path of length 2 or a pair of parallel edges. Therefore, $\bigcup_{i \in S} M_i$ can be decomposed into at most $\frac{\Delta+\mu}{2}$ paths and cycles of length 2. For each distinct $i, j \in T$, since $|M_i|$ and $|M_j|$ are odd and at most one of $|M_i|$ and $|M_j|$ is equal to 1, we can find vertex-disjoint $e_i \in M_i$ and $e_j \in M_j$, and thus decompose $M_i \cup M_j$ into at most 3 matchings of even size: $M_i \setminus e_i, M_j \setminus e_j$, and $\{e_i, e_j\}$. Thus $\bigcup_{i \in [r] \setminus S} M_i$ can be decomposed into at most $\frac{3(\Delta+\mu)}{2}$ matchings of even size. \square

Proof of Lemma 4.16. Start with $\mathcal{C} := \emptyset$. Note that by (i) and (ii), R has $c \leq \beta^{-1}$ connected components C_1, \dots, C_c . For each $i \in [c]$, colour each $x \in V_\Gamma(C_i)$ with colour i . We say a path in some \mathcal{P}_j is *monochromatic* if its endpoints are coloured with the same colour, and *bichromatic* otherwise. We say a monochromatic path is *coloured i* if its endpoints are coloured i , and we say a bichromatic path is *coloured with $\{i, i'\}$* if one of its endpoint is coloured i and the other is coloured i' . Observe that exceptional vertices are left uncoloured but, by (iii), all paths in $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell$ have coloured endpoints. A path of length 2 with internal vertex in V_0 is called an *exceptional path*.

By (vii'), for each $i \in [\ell]$, $|\mathcal{P}_i| \leq c \leq \beta^{-1}$. Moreover, (iii) and (xi) imply that no path in $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell$ contains an edge inside V_0 . Thus, by repeatedly taking maximal monochromatic subpaths, each path in $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell$ admits a decomposition $\mathcal{D}_{\text{mono}} \cup \mathcal{D}_{\text{bi}}$, where $\mathcal{D}_{\text{mono}}$ is a set of at most $c \leq \beta^{-1}$ monochromatic paths of distinct colours and \mathcal{D}_{bi} is a set of bichromatic edges and exceptional paths such that, if $P, P' \in \mathcal{D}_{\text{bi}}$ are distinct, then they are coloured with distinct pairs of colours. This induces a decomposition of $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell$ into

- $\ell' \leq \beta^{-2} \ell$ monochromatic subpaths $P_1, \dots, P_{\ell'}$; and

- for each $1 \leq i < i' \leq c$, a set $\mathcal{Q}_{ii'}$ of at most $\beta^{-1}\ell$ bichromatic edges and exceptional paths coloured with $\{i, i'\}$.

Each monochromatic path P_i will be tied into a cycle. In order to cover the bichromatic paths, we partition each $\mathcal{Q}_{ii'}$ as follows. Observe that, by (xi), the following holds.

(†) For any $1 \leq i < i' \leq c$, the exceptional paths in $\mathcal{Q}_{ii'}$ have distinct internal vertices.

By removing at most one edge or exceptional path from each $\mathcal{Q}_{ii'}$, we may assume $|\mathcal{Q}_{ii'}|$ is even for any $1 \leq i < i' \leq c$. Let E' be the set of deleted edges. Then, $|E'| \leq \beta^{-2}$, as desired for (a).

Let $1 \leq i < i' \leq c$. Define a multiset $\mathcal{Q}_{ii'}^*$ of bicoloured edges coloured with $\{i, i'\}$ by replacing each exceptional path in $\mathcal{Q}_{ii'}$ by a *fictive edge* between its endpoints. Since $|V_0| \leq \varepsilon n$, each edge in $\mathcal{Q}_{ii'}^*$ has multiplicity at most $\varepsilon n + 1$. Then, by (v'), we can apply Lemma 4.18 with $\mathcal{Q}_{ii'}^*$ playing the role of G , $\Delta \leq \varepsilon n$, and $\mu \leq \varepsilon n + 1$ to obtain $\ell_{ii'}^* \leq 4\varepsilon n$ matchings of even size, $\ell_{ii'}' \leq 2\varepsilon n$ monochromatic paths of length 2, and $\ell_{ii'}'' \leq 2\varepsilon n$ cycles of length 2. Denote the matchings by $M_{ii's}^*$, with $s \in [\ell_{ii'}^*]$. Replace, in the paths and cycles of length 2, the fictive edges by their corresponding exceptional paths. By (†), we thus obtain $\ell_{ii'}$ edge-disjoint monochromatic paths, which we denote by $P_{ii's}$, with $s \in [\ell_{ii'}]$, and $\ell_{ii'}'$ edge-disjoint cycles which we add to \mathcal{C} . Note that $|\mathcal{C}| = \sum_{1 \leq i < i' \leq c} \ell_{ii'}' \leq \sqrt{\varepsilon} n$. Each monochromatic path $P_{ii's}$ will be tied into a cycle.

For each $1 \leq i < i' \leq c$ and $j \in [\ell_{ii'}^*]$, if there exists $j' \in [k]$ such that $|V(M_{ii'j}^*) \cap V_{j'}| > \zeta m$, then randomly partition $M_{ii'j}^*$ into $2\zeta^{-1}$ submatchings whose sizes are even and approximately equal. By Lemma 3.1, we may assume that each of the submatchings obtained contains at most ζm edges with an endpoint in $V_{j'}$, for each $j' \in [k]$. Denote by $\mathcal{Q}_{ii'j}$, with $j \in [\ell_{ii'}'']$, the $\ell_{ii'}'' \leq \frac{8\varepsilon n}{\zeta}$ sets of paths obtained from these submatchings by replacing the fictive edges by their corresponding exceptional paths. By construction and (†), the following hold.

- (1) For any $j' \in [k]$, $|V(\mathcal{Q}_{ii'j}) \cap V_{j'}| \leq \zeta m$.
- (2) $|\mathcal{Q}_{ii'j}|$ is even.
- (3) All paths in $\mathcal{Q}_{ii'j}$ are pairwise vertex-disjoint, bichromatic, and coloured with $\{i, i'\}$.
- (4) The paths in $\mathcal{Q}_{ii's}$, for all $s \in [\ell_{ii'}'']$, and the paths $P_{ii's}$, for all $s \in [\ell_{ii'}]$, are all pairwise edge-disjoint.

Each set $\mathcal{Q}_{ii's}$ will be tied into a cycle.

We now aim to apply Lemmas 4.12 and 4.14. Note that all the edges that we still need to cover with cycles belong to one the monochromatic paths P_i , or one of the monochromatic paths $P_{ii's}$, or one of the sets $\mathcal{Q}_{ii's}$. As mentioned above, the goal is tie each of these into a cycle, i.e. the sets $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ in Lemmas 4.12 and 4.14 will consist of the sets of the form $\{P_i\}$, $\{P_{ii's}\}$, or $\mathcal{Q}_{ii's}$. Formally, proceed as follows. Let $\ell'' := \ell' + \sum_{1 \leq i < i' \leq c} (\ell_{ii'} + \ell_{ii'}'')$. Then, by (vi'), $\ell'' \leq \ell\beta^{-2} + \beta^{-2} \left(2\varepsilon n + \frac{8\varepsilon n}{\zeta} \right) \leq \beta^2 n$. Denote by $\mathcal{P}'_1, \dots, \mathcal{P}'_{\ell''}$ the sets in

$$\{\{P_i\} \mid i \in [\ell']\} \cup \{\{P_{ii'j}\} \mid 1 \leq i < i' \leq c, j \in [\ell_{ii'}']\} \cup \{\mathcal{Q}_{ii'j} \mid 1 \leq i < i' \leq c, j \in [\ell_{ii'}'']\}.$$

By construction, we can successively apply Lemmas 4.12 and 4.14 to tie up the paths in each \mathcal{P}'_i into a cycle as follows. First, let $E_1, \dots, E_{\ell''}$ be the sets of edges of Γ obtained after applying Lemma 4.12 with $\mathcal{P}'_1, \dots, \mathcal{P}'_{\ell''}$, and ℓ'' playing the roles of $\mathcal{P}_1, \dots, \mathcal{P}_\ell$, and ℓ , respectively. Let $\mathcal{Q}_1, \dots, \mathcal{Q}_{\ell''}$ be the sets of paths as in part (a) of Lemma 4.12. Note that, for any $i \in [\ell'']$ and $j \in [k]$, by part (c) of Lemma 4.12, $|V(\mathcal{Q}_i) \cap V_j| \leq \sqrt{\zeta} m$. Moreover, condition (viii) of Lemma 4.14 holds for the sets $\mathcal{Q}_1, \dots, \mathcal{Q}_{\ell''}$ since, by construction, each \mathcal{P}'_i either contains a single monochromatic path or, an even number of bichromatic paths coloured with the same pair of colours. Let $\Gamma' := \Gamma \setminus (E_1 \cup \dots \cup E_{\ell''})$ and note that, by Lemma 4.2 and part (b) of Lemma 4.12, V_0, V_1, \dots, V_k is an $(\varepsilon^{\frac{1}{9}}, \beta, k, m, R)$ -superregular partition of Γ' . Thus, we can now apply Lemma 4.14 with $\mathcal{Q}_1, \dots, \mathcal{Q}_{\ell''}, \ell'', \Gamma', \varepsilon^{\frac{1}{9}}$, and $\sqrt{\zeta}$ playing the roles of $\mathcal{P}_1, \dots, \mathcal{P}_\ell, \ell, \Gamma, \varepsilon$, and ζ , respectively. Add all cycles obtained to \mathcal{C} and note that $|\mathcal{C}| \leq \sqrt{\varepsilon} n + \beta^2 n \leq \beta n$. Denote by $E'_1, \dots, E'_{\ell''}$ the sets of edges of Γ' obtained. Define $E := E_1 \cup \dots \cup E_{\ell''} \cup E'_1 \cup \dots \cup E'_{\ell''}$ and

observe that property (b) of Lemma 4.16 holds by part (b) of Lemmas 4.12 and 4.14. This completes the proof. \square

Finally, in Step 5 of the proof of Theorem 1.10(ii) (see the proof overview), we will need to cover a few excess edges. This will be achieved using Lemma 4.19. The idea is similar to the approach described in Figure 1. An even cycle in the reduced graph can be decomposed into two matchings M and M' . We can then use a few of the edges of the pairs in M' to tie together a path from each pair in M , and similarly for M and M' exchanged. More precisely, we prove the following.

Lemma 4.19. *Let $0 < \frac{1}{m} \ll \varepsilon, \zeta \ll d \leq 1$ and suppose $k \in \mathbb{N}^*$ is even. Let G be a graph on vertex set V and V_1, \dots, V_k be a partition of V into k clusters of size m . Suppose that for any $i \in [k]$, the pair $G[V_i, V_{i+1}]$ is $[\varepsilon, d]$ -superregular (where $V_{k+1} := V_1$). Suppose that $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ are sets of paths on V satisfying the following.*

- (i) $\ell \leq \zeta m$.
- (ii) *For each $i \in [\ell]$, there exists $I_i \subseteq [k]_{\text{odd}}$ or $I_i \subseteq [k]_{\text{even}}$ such that we can write $\mathcal{P}_i = \{P_{i,j} \mid j \in I_i\}$, where, for each $j \in I_i$, $P_{i,j}$ is a path of length at most $\frac{dm}{10}$ with an endpoint in V_j , an endpoint in V_{j+1} and $V(P_{i,j}) \subseteq V_j \cup V_{j+1}$.*
- (iii) *Any vertex $x \in V$ is an endpoint of at most 4 paths in $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_\ell$.*

Then there exist disjoint $E_1, \dots, E_\ell \subseteq E(G)$ such that the following hold.

- (a) *Each $x \in V$ is an endpoint of at most 6 edges in $E_1 \cup \dots \cup E_\ell$.*
- (b) *For each $i \in [\ell]$, $\mathcal{P}_i \cup E_i$ forms a cycle.*

Proof. Assume inductively that for some $0 \leq i \leq \ell$, we have constructed disjoint sets $E_1, \dots, E_i \subseteq E(G)$ such that the following hold.

- (1) For each $j \in [i]$, $\mathcal{P}_j \cup E_j$ forms a cycle C_j .
- (2) Each $x \in V$ is an internal vertex of at most one link path in $C_1 \cup \dots \cup C_i$.
- (3) Let $j \in [i]$ and $i_1 < \dots < i_s$ be an enumeration of I_j . Let Q be a link path in C_j . Then there exists $t \in [s]$ such that the following hold. The path Q links P_{j,i_t} and $P_{j,i_{t+1}}$, where $i_{s+1} := i_1$. Moreover, $V(Q) \subseteq V_{i_t+1} \cup \dots \cup V_{i_{t+1}}$. Finally, Q contains at most 3 edges of $G[V_{i_t+1}, V_{i_t+2}]$ and at most one edge of $G[V_{j'}, V_{j'+1}]$ for each $j' = i_t+2, i_t+3, \dots, i_{t+1}-1$.

Observe that by (2) and (iii), a vertex $x \in V$ is an endpoint of at most 6 edges in $E_1 \cup \dots \cup E_i$. Thus, if $i = \ell$, we are done. We may therefore assume that $i < \ell$.

Let $i_1 < i_2 < \dots < i_s$ be an enumeration of I_{i+1} . For each $t \in [s]$, denote by x_{i_t} and $x_{i_{t+1}}$ the endpoints of P_{i+1} in V_{i_t} and $V_{i_{t+1}}$, respectively. Define $G' := (G \setminus \bigcup_{j \in [i]} E_j) - (V(\mathcal{P}_{i+1}) \cup V(\bigcup_{j \in [i]} E_j) \setminus \{x_{i_t}, x_{i_{t+1}} \mid t \in [s]\})$. For any $j \in [k]$, let V'_j be obtained from V_j by removing the vertices in $V(\mathcal{P}_{i+1}) \cup V(E_1 \cup \dots \cup E_i) \setminus \{x_{i_t}, x_{i_{t+1}} \mid t \in [s]\}$ and note that, by (3) and (iii), $|V_j \setminus V'_j| \leq \frac{dm}{10} + 2\ell \leq \frac{dm}{5}$. Thus, by Lemmas 4.1 and 4.2, $G'[V'_j, V'_{j+1}]$ is $\varepsilon^{\frac{1}{3}}$ -regular. Moreover, each $x \in V'_j$ satisfies $|N_{G'}(x) \cap V'_{j+1}| \geq (d - \varepsilon)m - 6 - \frac{dm}{5} \geq \frac{3d}{5}|V'_{j+1}|$ and, similarly, each $x' \in V'_{j+1}$ satisfies $|N_{G'}(x') \cap V'_j| \geq \frac{3d}{5}|V'_j|$.

Then, for each $t \in [s]$, we find an $(x_{i_{t+1}}, x_{i_{t+1}})$ -path Q_{i_t} in G' as follows. First, we find a path $Q'_{i_t} = x_{i_{t+2}} \dots x_{i_{t+1}}$ in G' where $x_j \in V'_j$ for each $j = i_t + 2, \dots, i_{t+1}$. Then, we apply Proposition 4.4 (with $G'[V'_{i_{t+1}}, V'_{i_{t+2}}]$, $\varepsilon^{\frac{1}{3}}$, and $\frac{3d}{5}$ playing the roles of G, ε , and d) to find an $(x_{i_{t+1}}, x_{i_{t+2}})$ -path Q''_{i_t} of length at most 3 in $G'[V'_{i_{t+1}}, V'_{i_{t+2}}]$. Let $Q_{i_t} := x_{i_{t+1}} Q'_{i_t} x_{i_{t+2}} Q''_{i_t} x_{i_{t+1}}$. Setting $E_{i+1} := \bigcup_{t \in [s]} E(Q_{i_t})$ completes the proof. \square

4.3. Making superregular pairs Eulerian and regular. As discussed in the proof overview, in Step 5 of the proof of Theorem 1.10(ii), we will need to decompose superregular pairs into Hamilton cycles. Thus we will need to ensure that our superregular pairs are Eulerian and regular. In this section, we introduce efficient tools for achieving this.

Lemma 4.20. *Let $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll d \leq 1$. Then there exists a constant $c = c(d, k)$ such that the following holds. Let G be an Eulerian graph and V_0, V_1, \dots, V_k be an $(\varepsilon, \geq d, k, m, m', R)$ -superregular equalised partition of G . Suppose that V_0 is a set of isolated vertices in G . Then, there exists a spanning subgraph $G' \subseteq G$ such that the following hold. For any $ij \in E(R)$, $G'[V_i, V_j]$ is Eulerian. Moreover, G' can be obtained from G by removing at most c edge-disjoint cycles. In particular, by Lemma 4.2, V_0, V_1, \dots, V_k is an $(2\sqrt{\varepsilon}, \geq d, k, m, m', R)$ -superregular equalised partition of G' .*

Proof. For simplicity, we assume that R is connected. If R is not connected, we can proceed similarly, but apply our arguments to each component of R separately. For any $AB \in E(R)$, we write A^B for the support cluster of A with respect to B . Let $H \subseteq G$, $i \in [k]$, and $x \in V_i$. We define the *oddy* of x in H , denoted $\mathcal{O}_H(x)$, as the number of indices $j \in [k]$ such that $|N_H(x) \cap V_j|$ is odd. The *oddy* of H is defined as $\mathcal{O}(H) := \sum_{x \in V(H)} \mathcal{O}_H(x)$. Let $S(H) := \{x \in V(H) \mid \mathcal{O}_H(x) > 0\}$ and $\mathcal{N}(H) := |S(H)|$. Thus, $G[V_i, V_j]$ is Eulerian for all $ij \in E(R)$ if and only if $\mathcal{N}(G) = 0$, or, equivalently, if and only if $\mathcal{O}(G) = 0$. Our argument relies on the two following observations:

- (i) any graph contains an even number of odd degree vertices, and,
- (ii) in an Eulerian graph, the oddity of each vertex is even.

Our proof splits into three steps. In Step 1, we significantly reduce the number of vertices of positive oddity by removing cycles of linear length. Then, in Step 2, we will proceed similarly but optimise the number of vertices whose oddity is reduced in order to decrease $\mathcal{N}(G)$ to a bounded number. Then, in Step 3, we will be able to use a greedy approach.

Step 1: Decreasing the number of vertices with positive oddity to fewer than $\frac{dm'}{2}$.

If $\mathcal{N}(G) < \frac{dm'}{2}$, let $G_1 := G$ and go to the next step. Otherwise, we claim that there exists $G_1 \subseteq G$ such that $\mathcal{N}(G_1) < \frac{dm'}{2}$ and G_1 can be obtained from G by removing at most $c_1 := \frac{20k^2}{d}$ cycles.

Consider the following algorithm. Pick $x_0 \in S(G)$ and let P_0 be the path x_0 of length 0. Suppose that after $i \geq 0$ steps, we have extended P_0 to an (x_0, x_i) -path P_i . Let A_i be the cluster such that $x_i \in A_i$. Let $G^i := G - (V(P_i) \setminus \{x_i\})$ and $G^{i,0} := G - (V(P_i) \setminus \{x_i, x_0\})$.

Case 1: P_i has length less than $\frac{dm'}{4}$.

- (a) If there exist a cluster B_i and a vertex $x_{i+1} \in (A_i \cup B_i) \cap (S(G) \setminus V(P_i))$ such that both x_i and x_{i+1} have odd degree in $(G \setminus P_i)[A_i, B_i]$, pick such B_i and x_{i+1} . Note that $|V(P_i)| \leq \frac{dm'}{4}$, so $\delta((G^i - V(P_i))[A_i^{B_i}, B_i^{A_i}]) \geq \frac{2dm'}{3}$. Moreover, by Lemma 4.1, $(G^i - V(P_i))[A_i^{B_i}, B_i^{A_i}]$ is $\sqrt{\varepsilon}$ -regular. Apply Proposition 4.4 (with $(G^i - V(P_i))[A_i^{B_i}, B_i^{A_i}]$, $\sqrt{\varepsilon}$, and $\frac{2d}{3}$ playing the roles of G, ε , and d) to find an (x_i, x_{i+1}) -path Q of length at most 4 in $G^i[A_i, B_i]$. Let $P_{i+1} := x_0 P_i x_i Q x_{i+1}$. Finally, observe that for any $x \in V(G)$,

$$(4.1a) \quad \mathcal{O}_{G \setminus P_{i+1}}(x) = \mathcal{O}_{G \setminus P_i}(x) - 1, \quad \text{if } x \in \{x_i, x_{i+1}\};$$

$$(4.1b) \quad \mathcal{O}_{G \setminus P_{i+1}}(x) = \mathcal{O}_{G \setminus P_i}(x), \quad \text{otherwise.}$$

- (b) Otherwise, pick any $x_{i+1} \in S(G) \setminus V(P_i)$. Let A_{i+1} denote the cluster which contains x_{i+1} . We claim that there exists an (x_i, x_{i+1}) -path Q of length at most $2k$ in $G^i - (V(P_i) \setminus \{x_i, x_{i+1}\})$. Indeed, observe that for any $UW \in E(R)$, $\delta((G^i - (V(P_i) \setminus \{x_i, x_{i+1}\}))[U, W]) \geq (d - \varepsilon)m' - \frac{dm'}{4} \geq \frac{2dm'}{3}$ and, by Lemma 4.1, $(G^i - (V(P_i) \setminus \{x_i, x_{i+1}\}))[U, W]$ is $\sqrt{\varepsilon}$ -regular. Thus, if $A_i = A_{i+1}$, we can let $U \in N_R(A_i)$ and apply Proposition 4.4 (with $(G^i - (V(P_i) \setminus \{x_i, x_{i+1}\}))[A_i, U]$, $\sqrt{\varepsilon}$, and $\frac{2d}{3}$ playing the roles of G, ε , and d) to obtain an (x_i, x_{i+1}) -path Q of length at most 4 in $G^i - (V(P_i) \setminus \{x_i, x_{i+1}\})$. Similarly, if $A_i A_{i+1} \in E(R)$, then we can apply Proposition 4.4 (with $(G^i - (V(P_i) \setminus \{x_i, x_{i+1}\}))[A_i, A_{i+1}]$, $\sqrt{\varepsilon}$, and $\frac{2d}{3}$ playing the roles of G, ε , and d) to obtain an (x_i, x_{i+1}) -path Q of length at most 4 in $G^i - (V(P_i) \setminus \{x_i, x_{i+1}\})$. Suppose that $A_i \neq A_{i+1}$ and $A_i A_{i+1} \notin E(R)$. Let Q' be an (A_i, A_{i+1}) -path in R . (This is possible since, by assumption, R is connected.) Note that Q' is a path of length at most $k - 1$. Denote $Q' = A_i U_1 \dots U_\ell A_{i+1}$. (Note that, by assumption, $\ell \geq 1$.) By the above,

there exists a path $x_i u_1 \dots u_\ell$ in $G^i - (V(P_i) \setminus \{x_i, x_{i+1}\})$ such that, for each $j \in [\ell]$, $u_j \in U_j$. Apply Proposition 4.4 (with $(G^i - (V(P_i) \setminus \{x_i, x_{i+1}\}))[U_\ell, A_{i+1}]$, $\sqrt{\varepsilon}$, and $\frac{2d}{3}$ playing the roles of G, ε , and d) to obtain a (u_ℓ, x_{i+1}) -path Q'' of length at most 4 in $(G^i - (V(P_i) \setminus \{x_i, x_{i+1}\}))[U_\ell, A_{i+1}]$. Then, $Q := x_i u_1 \dots u_\ell Q'' x_{i+1}$ is a path of length at most $(k-2) + 4 \leq 2k$ in $G^i - (V(P_i) \setminus \{x_i, x_{i+1}\})$, as desired.

Let $P_{i+1} := x_0 P_i x_i Q x_{i+1}$. Note that (ii) implies that there exists a cluster B_i such that x_i has odd degree in $(G \setminus P_i)[A_i, B_i]$. Thus, Case 1(b) can only occur at most $|E(R)| < k^2$ times in total. Finally, observe that for any $x \in V(G)$,

$$(4.2a) \quad \mathcal{O}_{G \setminus P_i}(x) - 1 \leq \mathcal{O}_{G \setminus P_{i+1}}(x) \leq \mathcal{O}_{G \setminus P_i}(x) + 1, \quad \text{if } x \in \{x_i, x_{i+1}\};$$

$$(4.2b) \quad \mathcal{O}_{G \setminus P_i}(x) - 2 \leq \mathcal{O}_{G \setminus P_{i+1}}(x) \leq \mathcal{O}_{G \setminus P_i}(x) + 2, \quad \text{if } x \in V(P_{i+1}) \setminus (V(P_i) \cup \{x_{i+1}\});$$

$$(4.2c) \quad \mathcal{O}_{G \setminus P_{i+1}}(x) = \mathcal{O}_{G \setminus P_i}(x), \quad \text{otherwise.}$$

Case 2: P_i has length at least $\frac{dm'}{4}$. Note that, by Case 1, P_i has length at most $\frac{dm'}{4} + 2k$. Thus, by similar arguments as above, there exists an (x_i, x_0) -path Q in $G^{i,0}$ of length at most $2k$. Output the cycle $C := x_0 P_i x_i Q x_0$ and observe that C has length at least $\frac{dm'}{4}$. Moreover, for any $x \in V(G)$,

$$(4.3a) \quad \mathcal{O}_{G \setminus P_i}(x) - 1 \leq \mathcal{O}_{G \setminus C}(x) \leq \mathcal{O}_{G \setminus P_i}(x) + 1, \quad \text{if } x \in \{x_i, x_0\};$$

$$(4.3b) \quad \mathcal{O}_{G \setminus P_i}(x) - 2 \leq \mathcal{O}_{G \setminus C}(x) \leq \mathcal{O}_{G \setminus P_i}(x) + 2, \quad \text{if } x \in V(C) \setminus V(P_i);$$

$$(4.3c) \quad \mathcal{O}_{G \setminus C}(x) = \mathcal{O}_{G \setminus P_i}(x), \quad \text{otherwise.}$$

We claim that $\mathcal{O}(G \setminus C) \leq \mathcal{O}(G) - \frac{dm'}{20}$. Indeed, as observed above, Case 1(b) can only occur fewer than k^2 times, and, clearly, Case 2 can only occur at most once. Thus, Case 1(a) occurs at least $\frac{dm'}{32}$ times and, therefore, (4.1)–(4.3) imply $\mathcal{O}(G \setminus C) \leq \mathcal{O}(G) - 2 \cdot \frac{dm'}{32} + 2(2k+1)k^2 \leq \mathcal{O}(G) - \frac{dm'}{20}$.

If $\mathcal{N}(G \setminus C) < \frac{dm'}{2}$, let $G_1 := G \setminus C$. Otherwise, repeatedly run the algorithm (where, in each iteration, the current graph plays the role of G) and delete the resulting cycle until a graph G_1 with $\mathcal{N}(G_1) < \frac{dm'}{2}$ is obtained. Note that we need to run the algorithm and delete the cycle obtained at most $c_1 = \frac{20k^2}{d}$ times. Indeed, assume we repeatedly ran the algorithm and deleted the resulting cycle c_1 times and let G_1 be the graph obtained. First, observe that we have delete at most $2c_1$ edges incident to each vertex, so ε -regular pairs still have minimum degree at least $(d - 2\varepsilon)m'$ and, thus, in each iteration, the algorithm is always well defined. Since $\mathcal{O}(G) \leq m'k^2$, we have $\mathcal{O}(G_1) \leq m'k^2 - \frac{dm'}{20} \cdot \frac{20k^2}{d} < \frac{dm'}{2}$ and in particular $\mathcal{N}(G_1) < \frac{dm'}{2}$. Thus G_1 can be obtained from G by removing at most c_1 cycles, as desired.

Step 2: Decreasing the number of vertices of positive oddity to fewer than $100k^4$. If $\mathcal{N}(G_1) < 100k^4$, let $G_2 := G_1$ and go to the next step. Otherwise, we claim that there exists $G_2 \subseteq G$ such that G_2 can be obtained from G_1 by removing at most $c_2 := \frac{21k}{2}$ cycles and such that $\mathcal{N}(G_2) < 100k^4$.

We proceed similarly as above, but since the number of vertices of positive oddity has now been significantly reduced, we can proceed more carefully. Indeed, we observe that, in the above algorithm, oddity may be created whenever Case 1(b) occurs (as well as in Case 2). Note that Case 1(b) occurs at stage i if, for all B_i as in Case 1(a), all vertices of odd degree in $G_1[A_i, B_i]$ already belong to $V(P_i)$. Thus, in order to make our algorithm more efficient, we shall add the extra condition that the internal vertices of the short path used to extend the paths P_i have oddity 0. Namely, we now let $G^i := G_1 - ((S(G_1) \cup V(P_i)) \setminus \{x_i\})$ and $G^{i,0} := G_1 - ((S(G_1) \cup V(P_i)) \setminus \{x_i, x_0\})$. We note that this improvement could not have been implemented in Step 1 since $\mathcal{N}(G)$ was large. Moreover, we observe that Case 1(b) may still occur. We proceed as in Case 1 of Step 1 if P_i has length less than $\frac{dm'}{4}$ and $S(G_1) \not\subseteq V(P_i)$ and as in Case 2 of Step 1 if P_i has length at least $\frac{dm'}{4}$ (Case 2(a)) or $S(G_1) \subseteq V(P_i)$ (Case 2(b)), with G_1 playing the role of G .

By similar arguments as above, $|S(G_1)| \leq \frac{5dm'}{8}$ implies the desired short paths always exist and so the algorithm is well defined. Using similar arguments as in Step 1, one can show that $\mathcal{N}(G_1 \setminus C) < \mathcal{N}(G_1) + (2k+1)k^2$. Moreover, if the algorithm terminates in Case 2(a) (i.e. if P_i has length at least $\frac{dm'}{4}$), then, as before, $\mathcal{O}(G_1 \setminus C) \leq \mathcal{O}(G_1) - \frac{dm'}{20}$. If the algorithm terminates in Case 2(b) (i.e. if $S(G_1) \subseteq V(C)$), then we note that, since Cases 1(b) and 2 occur at most k^2 times in total, by (4.1a), (4.2a), and (4.3a), we have $\mathcal{O}_{G_1 \setminus C}(x) = \mathcal{O}_{G_1}(x) - 2$ for all but at most $2k^2$ vertices $x \in S(G_1)$.

If $\mathcal{N}(G_1 \setminus C) < 100k^4$, then let $G_2 := G_1 \setminus C$. Otherwise, repeatedly run the algorithm (where, in each iteration, the current graph plays the role of G_1) and delete the resulting cycle until a graph $G_2 \subseteq G_1$ with $\mathcal{N}(G_2) < 100k^4$ is obtained. We claim that we need to run the algorithm and delete the cycle obtained at most $c_2 = \frac{21k}{2}$ times. Indeed, assume we ran the algorithm and deleted the resulting cycle c_2 times and let G_2 be the graph obtained. Note that, in each iteration of the algorithm, the current graph has fewer than $\frac{dm'}{2} + 5c_2k^3 \leq \frac{5dm'}{8}$ vertices of positive oddity, so the algorithm is well defined in each of the iterations. If the algorithm terminates in Case 2(b) in at least $\frac{k}{2}$ of the iterations, then we note that all but at most $2c_2k^2$ of the vertices in $S(G_1)$ now have oddity 0. Therefore, $\mathcal{N}(G_2) \leq 2c_2k^2 + 5c_2k^3 < 100k^4$, as desired. Otherwise, the algorithm terminates in Case 2(a) in at least $10k$ of the iterations. Therefore, $\mathcal{O}(G_2) \leq \mathcal{O}(G_1) - 10k \cdot \frac{dm'}{20} \leq 0$ and so $\mathcal{N}(G_2) = 0$. Thus G_2 can be obtained from G_1 by removing at most c_2 cycles.

Step 3: Removing all oddity. If $\mathcal{N}(G_2) = 0$, we set $G' := G_2$. Otherwise, we claim that there exists $G' \subseteq G_2$ such that G' can be obtained from G_2 by removing at most $c_3 := 25k^4(k-1)$ cycles and such that $\mathcal{N}(G') = 0$.

Consider the following algorithm. Pick a vertex $x_0 \in S(G_2)$ and let P_0 be the path x_0 of length 0. Suppose that after $|S(G_2)| \geq i \geq 0$ steps we have extended P_0 to an (x_0, x_i) -path P_i of length at most $4i$ such that $x_i \in S(G_2)$, $\mathcal{O}_{G_2 \setminus P_i}(x_i) = \mathcal{O}_{G_2}(x_i) - 1$ and $\mathcal{O}_{G_2 \setminus P_i}(x) \leq \mathcal{O}_{G_2}(x)$ for all $x \in V(G) \setminus V_0$. Denote by A_i the cluster such that $x_i \in A_i$. Let $x_{i+1} \in S(G_2) \setminus \{x_i\}$ be such that there exists a cluster $B_i \neq A_i$ such that both x_i and x_{i+1} have odd degree in $(G_2 \setminus P_i)[A_i, B_i]$. Observe that such cluster and vertex exist by (i) and (ii).

Since $|V(P_i) \cup S(G_2)| \leq 6|S(G_2)| \leq 600k^4$, Lemma 4.2 implies that $(G_2 - ((V(P_i) \cup S(G_2)) \setminus \{x_i, x_{i+1}\})) [A_i^{B_i}, B_i^{A_i}]$ is $[2\sqrt{\varepsilon}, \geq d]$ -superregular. Apply Proposition 4.4 (with $(G_2 - ((V(P_i) \cup S(G_2)) \setminus \{x_i, x_{i+1}\})) [A_i^{B_i}, B_i^{A_i}]$, $2\sqrt{\varepsilon}$, and $d - 2\sqrt{\varepsilon}$ playing the roles of G, ε , and d) to obtain an (x_i, x_{i+1}) -path Q_{i+1} of length at most 4 in $(G_2 - ((V(P_i) \cup S(G_2)) \setminus \{x_i, x_{i+1}\})) [A_i^{B_i}, B_i^{A_i}]$. Then,

- (1) if $x_{i+1} \in V(P_i)$, output the cycle $C := x_{i+1}P_ix_iQ_{i+1}x_{i+1}$;
- (2) if $x_{i+1} \notin V(P_i)$, let $P_{i+1} := x_0P_ix_iQ_{i+1}x_{i+1}$.

Note that if $i = |S(G_2)|$, then $S(G_2) \subseteq V(P_i)$. Thus, if we are in case (2), then $i+1 \leq |S(G_2)|$, as desired.

Clearly, this algorithm eventually terminates, and, for each $x \in V(G)$, we have

$$\mathcal{O}_{G_2 \setminus C}(x) = \begin{cases} \mathcal{O}_{G_2}(x) - 2, & \text{if } x \in V(C) \cap S(G_2); \\ \mathcal{O}_{G_2}(x), & \text{otherwise.} \end{cases}$$

Moreover, $|V(C) \cap S(G_2)| \geq 2$ and, thus, $\mathcal{O}(G_2 \setminus C) \leq \mathcal{O}(G_2) - 4$.

If $\mathcal{N}(G_2 \setminus C) = 0$, then let $G' := G_2 \setminus C$. Otherwise, repeatedly run the algorithm (where, in each iteration, the current graph plays the role of G_2) and delete the resulting cycle until a graph G' with $\mathcal{N}(G') = 0$ is obtained. By the above, we clearly need to run the algorithm and delete the cycle obtained at most $c_3 = 25k^4(k-1)$ times. Let $c := c_1 + c_2 + c_3$. This completes the proof. \square

To regularise an Eulerian ε -regular pair, we adapt an argument of [20]. The idea is to repeatedly remove cycles covering all vertices of maximum degree. By ensuring that each vertex

of minimum degree is covered by at most half of the cycles, we are able to regularise the pair by deleting only a few cycles.

Lemma 4.21. *Suppose $0 < \frac{1}{m} \ll \eta, \varepsilon \ll d \leq 1$ and let $\varepsilon' := \max\{2\sqrt{\varepsilon}, 4\sqrt{\eta}\}$. Let G be an Eulerian (ε, d) -regular bipartite graph on vertex classes A, B of size m . Let $\Theta := \Delta(G) - \delta(G)$ and suppose $\Theta \leq \eta m$. Then there exists a spanning subgraph $H \subseteq G$ such that H is regular, ε' -regular, and can be obtained from G by removing at most 2Θ edge-disjoint cycles of length at least $\frac{2m}{3}$. In particular, H is r -regular for some $r \geq \Delta(G) - 4\Theta$.*

Proof. First note that $\delta(G) \geq \frac{dm}{2}$. Let $G_0 := G$. We proceed inductively to build

- spanning subgraphs $G_1 \supseteq G_2 \supseteq \dots \supseteq G_\ell$ of G ;
- sets of vertices $A_0^\Delta \subseteq A_1^\Delta \subseteq \dots \subseteq A_{\ell-1}^\Delta \subseteq A$ and $B_0^\Delta \subseteq B_1^\Delta \subseteq \dots \subseteq B_{\ell-1}^\Delta \subseteq B$;
- sets of vertices $A_j^{\delta,1}, A_j^{\delta,2} \subseteq A$ and $B_j^{\delta,1}, B_j^{\delta,2} \subseteq B$ for each even $j \in \{0, 1, \dots, \ell-1\}$;
- sets of vertices $S_j^A \subseteq A$ and $S_j^B \subseteq B$ for each $j \in \{0, 1, \dots, \ell-1\}$; and
- edge-disjoint cycles $C_0, C_1, \dots, C_{\ell-1}$;

such that G_ℓ is regular, $\ell \leq 2\Theta$, and, for each $i \in \{0, 1, \dots, \ell-1\}$, the following hold.

- (i) $A_i^\Delta = \{a \in A \mid d_{G_i}(a) = \Delta(G_i)\}$ and $B_i^\Delta = \{b \in B \mid d_{G_i}(b) = \Delta(G_i)\}$.
- (ii) If i is even, $A_i^{\delta,1}$ and $A_i^{\delta,2}$ are disjoint and such that $A_i^{\delta,1} \cup A_i^{\delta,2} = \{a \in A \mid d_{G_i}(a) = \delta(G_i)\}$, and, similarly, $B_i^{\delta,1}$ and $B_i^{\delta,2}$ are disjoint and such that $B_i^{\delta,1} \cup B_i^{\delta,2} = \{b \in B \mid d_{G_i}(b) = \delta(G_i)\}$.
- (iii) If i is even, then $A_i^\Delta \subseteq S_i^A \subseteq A \setminus A_i^{\delta,1}$, $B_i^\Delta \subseteq S_i^B \subseteq B \setminus B_i^{\delta,1}$ and $|S_i^A| = |S_i^B| \geq \frac{m}{3}$.
- (iv) If i is odd, then $A_i^\Delta \subseteq S_i^A \subseteq A \setminus (A_{i-1}^{\delta,2} \cap S_{i-1}^A)$, $B_i^\Delta \subseteq S_i^B \subseteq B \setminus (B_{i-1}^{\delta,2} \cap S_{i-1}^B)$ and $|S_i^A| = |S_i^B| \geq \frac{m}{3}$.
- (v) C_i is a Hamilton cycle of $G_i[S_i^A \cup S_i^B]$.
- (vi) $G_{i+1} = G_i \setminus C_i$.

Assume that for some even $0 \leq i \leq 2\Theta$, we have already constructed subgraphs G_j for each $j \in [i]$, sets A_j^Δ and B_j^Δ for each $j \in \{0, 1, \dots, i-1\}$, sets $A_j^{\delta,1}, A_j^{\delta,2}$ and $B_j^{\delta,1}, B_j^{\delta,2}$ for each even $j \in \{0, 2, \dots, i-2\}$, sets S_j^A and S_j^B for each $j \in \{0, 1, \dots, i-1\}$, and cycles C_j for each $j \in \{0, \dots, i-1\}$ such that (i)–(vi) are satisfied with j playing the role of i for all $0 \leq j \leq i$. If G_i is regular, let $\ell := i$. Otherwise, proceed as follows.

Let $A_i^\Delta := \{a \in A \mid d_{G_i}(a) = \Delta(G_i)\}$ and $B_i^\Delta := \{b \in B \mid d_{G_i}(b) = \Delta(G_i)\}$, so that (i) is satisfied for i . Also define $A_i^\delta := \{a \in A \mid d_{G_i}(a) = \delta(G_i)\}$ and $B_i^\delta := \{b \in B \mid d_{G_i}(b) = \delta(G_i)\}$. We will now construct $A_i^{\delta,1}, A_i^{\delta,2}$ and $B_i^{\delta,1}, B_i^{\delta,2}$, but, first note that, by (ii)–(vi), $\delta(G_i) \geq \delta(G_0) - i \geq \frac{dm}{2} - 2\eta m \geq \frac{dm}{3}$. A simple application of Lemma 3.1 shows that there exists a partition $A_i^{\delta,1} \cup A_i^{\delta,2}$ of A_i^δ such that all of the following hold.

- (a) If $|A_i^\delta| \geq \frac{dm}{6}$, then $\frac{|A_i^\delta|}{3} \leq |A_i^{\delta,2}| \leq |A_i^{\delta,1}| \leq \frac{2|A_i^\delta|}{3}$.
- (b) If $|A_i^\delta| < \frac{dm}{6}$, then $|A_i^{\delta,1}| = \left\lceil \frac{|A_i^\delta|}{2} \right\rceil$ and $|A_i^{\delta,2}| = \left\lfloor \frac{|A_i^\delta|}{2} \right\rfloor$.
- (c) For any $b \in B$, $|N_{G_i}(b) \setminus A_i^{\delta,1}|, |N_{G_i}(b) \setminus A_i^{\delta,2}| \geq \frac{dm}{12}$.

Similarly, there exists a partition $B_i^{\delta,1} \cup B_i^{\delta,2}$ of B_i^δ satisfying analogous properties. Note that, in particular, (ii) holds for i . We will now construct sets S_i^A, S_i^B such that (iii) is satisfied and $G_i[S_i^A \cup S_i^B]$ is Hamiltonian (in order to satisfy (v)).

We may assume without loss of generality that $|A \setminus A_i^{\delta,1}| \geq |B \setminus B_i^{\delta,1}|$ (the other case is similar). Clearly, $|A_i^\Delta| \leq m - |B_i^\delta|$. Thus,

$$(4.4) \quad m - |B_i^{\delta,1}| - |A_i^\Delta| \geq |B_i^{\delta,2}|.$$

Let $S_i^B := B \setminus B_i^{\delta,1}$ and $T_i^A \subseteq A \setminus (A_i^{\delta,1} \cup A_i^\Delta)$ be a set of size $|B \setminus B_i^{\delta,1}| - |A_i^\Delta|$ chosen uniformly at random. Let $S_i^A := T_i^A \cup A_i^\Delta$. By construction, (iii) holds for i . Thus, by (vi), Lemmas 4.1 and 4.2 imply that $G_i[S_i^A \cup S_i^B]$ is $3\varepsilon'$ -regular.

Claim 1. *With positive probability, $G_i[S_i^A \cup S_i^B]$ is Hamiltonian.*

Proof of Claim. By Lemma 4.5, it suffices to show that $\delta(G_i[S_i^A \cup S_i^B]) \geq \frac{d^2 m}{10^3}$ with positive probability. By construction, for any $a \in S_i^A$, we have $d_{S_i^B}(a) \geq \frac{dm}{12}$, as needed. It remains to show that, with high probability, $d_{S_i^A}(b) \geq \frac{d^2 m}{10^3}$ for all $b \in S_i^B$. If $|B_i^{\delta,1}| - |A_i^{\delta,1}| \leq \frac{dm}{24}$, then $|A \setminus (A_i^{\delta,1} \cup A_i^\Delta)| \leq \frac{dm}{24}$ and thus, by (c), $d_{S_i^A}(b) \geq \frac{dm}{24}$ for all $b \in S_i^B$.

We may therefore assume that $|B_i^{\delta,1}| - |A_i^{\delta,1}| \geq \frac{dm}{24}$. Then, $|B_i^{\delta,1}| \geq \frac{dm}{24}$, and, by (a) and (b), $|B_i^{\delta,2}| \geq \frac{|B_i^{\delta,1}|}{2} \geq \frac{dm}{48}$. Let $b \in S_i^B$. Then,

$$\mathbb{E} [d_{S_i^A}(b)] \geq \mathbb{E} [d_{T_i^A}(b)] + d_{A_i^\Delta}(b) \stackrel{(c), (4.4)}{\geq} \frac{d^2 m}{800}.$$

Thus, by Lemma 3.1, $\mathbb{P} \left[d_{S_i^A}(b) < \frac{d^2 m}{10^3} \right] \leq \frac{1}{m^2}$, and, a union bound over all $b \in S_i^B$ gives that, with positive probability, $d_{S_i^A}(b) \geq \frac{d^2 m}{10^3}$ for all $b \in S_i^B$. \diamond

Thus, we can let C_i be a Hamilton cycle of $G_i[S_i^A \cup S_i^B]$ and $G_{i+1} := G_i \setminus C_i$. Then, (v) and (vi) are satisfied for i .

If G_{i+1} is regular, let $\ell := i + 1$. Otherwise, proceed as follows. Let $A_{i+1}^\Delta := \{a \in A \mid d_{G_{i+1}}(a) = \Delta(G_{i+1})\}$, $B_{i+1}^\Delta := \{b \in B \mid d_{G_{i+1}}(b) = \Delta(G_{i+1})\}$, so that (i) is satisfied for $i + 1$. We will now proceed similarly as above to construct $S_{i+1}^A, S_{i+1}^B, C_{i+1}$ and G_{i+2} .

If $\Delta(G_i) - \delta(G_i) = 2$, we have $|A_i^\Delta| = |B_i^\Delta|$, $|A_i^\delta| = |B_i^\delta|$, $A = A_i^\delta \cup A_i^\Delta$, and $B = B_i^\delta \cup B_i^\Delta$. Then, by construction of G_{i+1} , we have $A_{i+1}^\Delta = A_i^\Delta \cup A_i^{\delta,1} \cup (A_i^{\delta,2} \setminus S_i^A)$ and $B_{i+1}^\Delta = B_i^\Delta \cup B_i^{\delta,1}$. Moreover, all vertices of $A \setminus A_{i+1}^\Delta$ and $B \setminus B_{i+1}^\Delta$ are vertices of minimum degree in G_{i+1} . Thus we can let $S_{i+1}^A := A_{i+1}^\Delta$ and $S_{i+1}^B := B_{i+1}^\Delta$ in order to satisfy (iv). We can then proceed as above to define C_{i+1} and G_{i+2} satisfying (v) and (vi) (and, in particular, G_{i+2} is regular). We may therefore assume that $\Delta(G_i) - \delta(G_i) > 2$. Note that

$$(4.5) \quad m \geq \begin{cases} |B_i^\delta| + |A_{i+1}^\Delta|, & \text{if } |A \setminus A_i^{\delta,2}| \geq |B \setminus B_i^{\delta,2}|, \\ |A_i^\delta| + |B_{i+1}^\Delta|, & \text{otherwise.} \end{cases}$$

We construct $S_{i+1}^A, S_{i+1}^B, C_{i+1}$, and G_{i+2} similarly as above, but, we now let $A_{i+1}^\Delta \subseteq S_{i+1}^A \subseteq A \setminus A_i^{\delta,2}$, $B_{i+1}^\Delta \subseteq S_{i+1}^B \subseteq B \setminus B_i^{\delta,2}$, and use (4.5) instead of (4.4).

We now show that we eventually obtain a regular graph G_ℓ , with $\ell \leq 2\Theta$. Assume i is even and G_i is not regular. By (i) and (iii)–(vi), $\Delta(G_i) = \Delta(G_0) - 2i$. Moreover, by (ii)–(vi), $\delta(G_i) \geq \delta(G_0) - i$. Thus,

$$0 < \Delta(G_i) - \delta(G_i) \leq (\Delta(G_0) - 2i) - (\delta(G_0) - i) \leq \Theta - i,$$

and, therefore, $i < \Theta$. Thus, $\ell \leq \Theta + 1 \leq 2\Theta$, as desired.

Let $H := G_\ell$. Clearly, H is regular. Moreover, $G - H = \bigcup_{i=0}^{\ell-1} C_i$, with $\ell \leq 2\Theta$, and so H can be obtained from G by removing at most 2Θ cycles. Moreover, by (iii)–(v), the cycles $C_0, \dots, C_{\ell-1}$ have length at least $\frac{2m}{3}$, as desired. Finally, by Lemma 4.2, H is ε' -regular. This completes the proof. \square

4.4. Robust decomposition lemma. Note that the contents of this section will only be used in Section 5.5 and so the reader may skip it and return to it later on.

A key tool in our proofs will be the robust decomposition lemma of [27], which implies the existence of a “robust” Hamilton decomposition of superregular pairs. More precisely, given a graph G consisting of suitable superregular pairs, it guarantees the existence of a spanning superregular graph G^{rob} such that $G^{\text{rob}} \cup H$ has a Hamilton decomposition for any very sparse H ,

i.e. G^{rob} is “robustly” Hamilton decomposable. (The graph H will be the “leftover” of an approximate decomposition of $G \setminus G^{\text{rob}}$.) Moreover, we can prescribe that a given Hamilton cycle in this decomposition contains a given set of edges. These edges will be the “fictive edges” discussed in the proof overview (see Figure 1 for example). To formalise the latter property, we need the notion of special path systems and special factors defined below. The fictive edges are extended into such special path systems prior to applying the robust decomposition lemma. It turns out to be more convenient to consider digraphs rather than graphs.

Given a digraph \vec{G} and a partition \mathcal{P} of $V(\vec{G})$ into k clusters V_1, \dots, V_k of equal size, a partition \mathcal{P}' of $V(\vec{G})$ is an ℓ -refinement of \mathcal{P} if \mathcal{P}' is obtained by splitting each V_i into ℓ subclusters of equal size. Let \vec{R} be the reduced digraph of \vec{G} with respect to \mathcal{P} and assume that for any $VW \in E(\vec{R})$, the pair $\vec{G}[V, W]$ is $[\varepsilon, d]$ -superregular. We say \mathcal{P}' is an ε -superregular ℓ -refinement of \mathcal{P} if the following holds. For any $V, W \in \mathcal{P}$ and $V', W' \in \mathcal{P}'$ with $V' \subseteq V$ and $W' \subseteq W$, if $VW \in E(\vec{R})$ then $\vec{G}[V', W']$ is $[\varepsilon, d]$ -superregular.

We say $(\vec{G}, \mathcal{P}, \mathcal{P}', \vec{R}, \vec{R}', C)$ is an $(\ell, k, m, \varepsilon, d)$ -bi-setup if the following properties are satisfied.

(BST1) \vec{G} is a directed graph.

(BST2) \mathcal{P} is a partition of $V(G)$ into k clusters of size m , where k is even, and, \vec{R} is the reduced digraph of \vec{G} with respect to \mathcal{P} .

(BST3) C is a Hamilton cycle of \vec{R} .

(BST4) \mathcal{P}' is an ℓ -refinement of \mathcal{P} , and, \vec{R}' is the reduced digraph of \vec{G} with respect to \mathcal{P}' .

(BST5) R and R' are complete balanced bipartite digraphs.

(BST6) For each $VW \in E(\vec{R}) \cup E(\vec{R}')$, the corresponding pair $\vec{G}[V, W]$ is $[\varepsilon, d]$ -superregular.

This is a special case of the setting in [27], which also requires the existence of a “universal walk” U in the reduced digraph \vec{R} . This is trivially implied by assumption (BST5).

Let \vec{G} be a digraph, \mathcal{P} be a partition of $V(\vec{G})$ into $2k$ clusters $A_1, \dots, A_k, B_1, \dots, B_k$ of size m , and \vec{R} be the corresponding reduced digraph of \vec{G} . Let $C := A_1 B_1 \dots A_k B_k$ be a Hamilton cycle of \vec{R} . Suppose $f \in \mathbb{N}$ divides k .

The canonical interval partition \mathcal{I} of C into f intervals consists of the intervals

$$I_i := A_{(i-1)\frac{k}{f}+1} B_{(i-1)\frac{k}{f}+1} A_{(i-1)\frac{k}{f}+2} \dots B_{i\frac{k}{f}} A_{i\frac{k}{f}+1}$$

for all $i \in [f]$, where $A_{k+1} := A_1$.

Let \mathcal{P}' be an ℓ -refinement of \mathcal{P} and for each $V \in \mathcal{P}$ denote by V^1, \dots, V^ℓ the partition of V induced by \mathcal{P}' . Suppose $\frac{2k}{f} \geq 3$. Let $i \in [f], h \in [\ell]$. Denote I_i by $V_{i,1} \dots V_{i,\frac{2k}{f}+1}$. A special path system SPS of style h in \vec{G} spanning the interval I_i consists of $\frac{2k}{f}$ matchings $M_1, \dots, M_{\frac{2k}{f}}$ such that the following hold.

(SPS1) For all $j \in [\frac{2k}{f}]$, M_j is a perfect matching between $V_{i,j}^h$ and $V_{i,j+1}^h$, with all edges oriented from $V_{i,j}^h$ to $V_{i,j+1}^h$.

(SPS2) SPS contains a fictive edge $f_{SPS} \in M_{\frac{2k}{f}-1}$ such that $E(SPS) \setminus \{f_{SPS}\} \subseteq E(\vec{G})$.

A special factor SF with parameters (ℓ, f) with respect to C and \mathcal{P}' in \vec{G} is a 1-regular digraph on $V(\vec{G})$ consisting of ℓf special path systems $SPS_{h,i}$, one for each $(h, i) \in [\ell] \times [f]$, where $SPS_{h,i}$ is a special path system of style h in \vec{G} spanning the interval I_i . We denote the set of fictive edges of SF by $\text{Fict}(SF) := \{f_{SPS_{h,i}} \mid h \in [\ell], i \in [f]\}$.

More generally, we will use the term *fictive edges* to refer to auxiliary edges that are artificially added to graphs. Whenever we add a set \mathcal{F} of fictive edges to a (di)graph G , we view them as being distinct from those in G , even if they create multiple edges. Similarly, we also allow multiple edges within \mathcal{F} and view these edges as being distinct from each other. We are now ready to state the (bipartite version of the) robust decomposition lemma. As indicated above, it

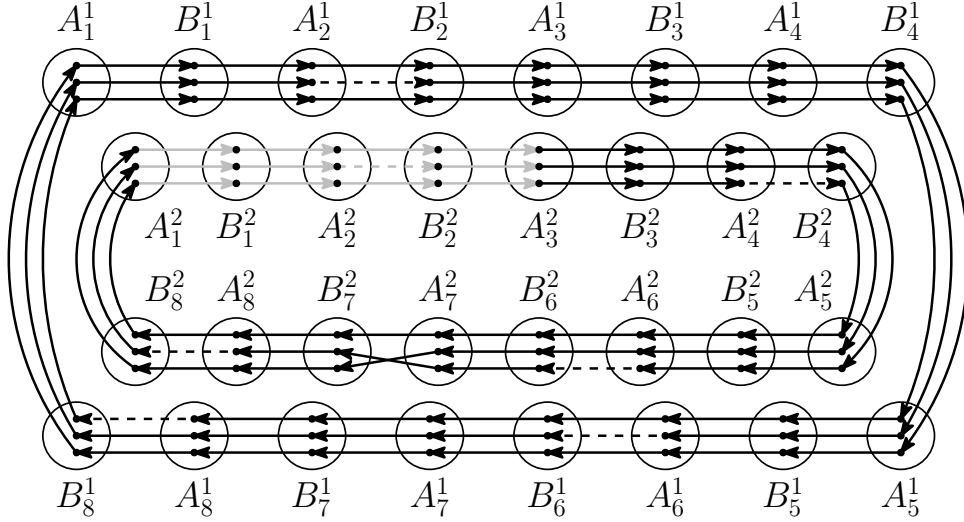


FIGURE 2. A special factor with parameters $(2,4)$ with respect to $C = A_1B_1A_2 \dots B_8$ and $\mathcal{P}' = \{A_1^1, A_1^2, A_2^1, \dots, A_8^2, B_1^1, \dots, B_8^2\}$. The fictive edges are represented by dashed edges. The gray edges form a special path system of style 2 spanning the first interval in the canonical interval partition of C into 4 intervals.

guarantees the existence of a robustly decomposable digraph \vec{G}^{rob} which consists of a “chord absorber” $\vec{CA}(r)$ and a “parity extended cycle absorber” $\vec{PCA}(r)$, as well as a prescribed set of special factors (which contain the fictive edges).

Lemma 4.22 (Robust Decomposition Lemma [27]). *Let $0 < \frac{1}{m} \ll \frac{1}{k} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{f} \ll \frac{r_1}{m} \ll d \ll \frac{1}{\ell}, \frac{1}{g} \ll 1$ and that $rk^2 \leq m$. Let*

$$r_2 := 96\ell g^2 kr, \quad r_3 := \frac{rfk}{q}, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3, \quad s := rfk + 7r^\diamond$$

and suppose that $\frac{k}{14}, \frac{k}{f}, \frac{k}{g}, \frac{q}{f}, \frac{m}{4\ell}, \frac{fm}{q}, \frac{2fk}{3g(g-1)}, \frac{\ell}{2} \in \mathbb{N}$. Let $(\vec{G}, \mathcal{P}, \mathcal{P}', \vec{R}, \vec{R}', C)$ be an $(\ell, k, m, \varepsilon, d)$ -bi-setup and $C = V_1 \dots V_k$. Suppose that \mathcal{P}^* is a $\frac{q}{f}$ -refinement of \mathcal{P} and that SF_1, \dots, SF_{r_3} are edge-disjoint special factors with parameters $(\frac{q}{f}, f)$ with respect to C and \mathcal{P}^* in \vec{G} . Let $SF := SF_1 \cup \dots \cup SF_{r_3}$. Then there exists a digraph $\vec{CA}(r)$ for which the following hold.

- (i) $\vec{CA}(r)$ is an $(r_1 + r_2)$ -regular spanning subdigraph of \vec{G} which is edge-disjoint from SF .
- (ii) Suppose that $SF'_1, \dots, SF'_{r^\diamond}$ are special factors with parameters $(1, 7)$ with respect to C and \mathcal{P} in \vec{G} which are edge-disjoint from each other and from $\vec{CA}(r) \cup SF$. Let $SF' := SF'_1 \cup \dots \cup SF'_{r^\diamond}$. Then there exists a digraph $\vec{PCA}(r)$ for which the following hold.
 - (a) $\vec{PCA}(r)$ is a $5r^\diamond$ -regular spanning subdigraph of \vec{G} which is edge-disjoint from $\vec{CA}(r) \cup SF \cup SF'$.
 - (b) Let \mathcal{SPS} be the set consisting of all the s special path systems contained in $SF \cup SF'$. Let V_{even} denote the union of all V_i over all $i \in [k]_{\text{even}}$ and define V_{odd} similarly. Suppose that \vec{H} is an r -regular bipartite digraph on $V(\vec{G})$ with vertex classes V_{even} and V_{odd} which is edge-disjoint from $\vec{G}^{\text{rob}} := \vec{CA}(r) \cup \vec{PCA}(r) \cup SF \cup SF'$. Then $\vec{H} \cup \vec{G}^{\text{rob}}$ has a decomposition into s edge-disjoint Hamilton cycles C_1, \dots, C_s . Moreover, C_i contains one of the special path systems from \mathcal{SPS} , for each $i \in [s]$.

5. PROOF OF THE MAIN THEOREMS

In Sections 5.1–5.5, we prove the main lemmas that will be needed for the proof of our theorems. These intermediate results are organised according to the structure of the proof overview. Theorems 1.10, 1.11, and 1.13 are proved in Section 5.6.

5.1. Applying Szemerédi’s regularity lemma and setting aside random subgraphs Γ and Γ' . This section corresponds to Step 1 of the proof overview. The proof of Lemma 5.1 relies of a straightforward application of Szemerédi’s regularity lemma and a cleaning procedure similar to the one used to prove the degree form of the regularity lemma. For details, see the appendix of the arXiv version.

Lemma 5.1. *Let $0 < \frac{1}{M} \ll \varepsilon \ll \zeta \ll d \ll \beta \ll \alpha \leq 1$ and $\frac{1}{M} \leq \frac{1}{L} \ll d$. Let $r \in \mathbb{N}^*$. Then there exist $M', n_0 \in \mathbb{N}^*$ such that the following holds. Let G be a graph on vertex set V with $|V| = n \geq n_0$, and $\delta(G) \geq \alpha n$. Then G can be decomposed into edge-disjoint graphs G', Γ, Γ', H , and, V can be partitioned into k clusters V_1, \dots, V_k of size m and an exceptional set V_0 such that the following properties are satisfied.*

- (i) $M \leq k \leq M'$.
- (ii) $\frac{m'}{r} \in \mathbb{N}^*$.
- (iii) V_0, V_1, \dots, V_k is an
 - $(\varepsilon, \geq d, k, m, m', R)$ -superregular equalised partition of G' ;
 - $(\varepsilon, \beta, k, m, m', R')$ -superregular equalised partition of Γ ;
 - $(\varepsilon, \zeta, k, m, m', R'')$ -superregular equalised partition of Γ' .
- (iv) R' and R'' are edge-disjoint and $R = R' \cup R''$.
- (v) G', Γ and, Γ' have the same support clusters.
- (vi) Each $x \in V \setminus V_0$ belongs to at least βk superregular pairs of Γ .
- (vii) There exists a decomposition $\mathcal{D}_{R'}$ of R' into at most $\frac{k}{2}$ cycles whose lengths are even and at least L and such that for any distinct $i, j, j' \in [k]$, if $V_j V_i V_{j'}$ is a subpath of a cycle in $\mathcal{D}_{R'}$ then the support clusters of V_i with respect to V_j and $V_{j'}$ are the same.
- (viii) V_0 is a set of isolated vertices in Γ, Γ' , and H .
- (ix) $\Delta(H) \leq 4dn$.

5.2. Covering the edges inside the exceptional set. This section corresponds to Step 2 of the proof overview.

Lemma 5.2. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \varepsilon \ll d \ll \beta \ll \alpha \leq 1$. Let G be a graph on vertex set V with $|V| = n$ and let Γ be edge-disjoint from G . Assume G and Γ satisfy the following.*

- (i) V_0, V_1, \dots, V_k is an $(\varepsilon, \beta, k, m, m', R')$ -superregular equalised partition of Γ .
- (ii) Any $x \in V \setminus V_0$ belongs to at least βk superregular pairs of Γ .
- (iii) V_0, V_1, \dots, V_k is an $(\varepsilon, \geq d, k, m, m', R)$ -superregular equalised partition of G .
- (iv) For any $x \in V_0$, $d_G(x) \geq \alpha n$.

Let $\varepsilon' := \varepsilon^{\frac{1}{73}}$. Then there exists $H \subseteq G \cup \Gamma$ such that the following hold.

- (a) $G[V_0] = H[V_0]$.
- (b) V_0, V_1, \dots, V_k is an $(\varepsilon', \geq d, k, m, m', R)$ -superregular equalised partition of $G \setminus H$.
- (c) V_0, V_1, \dots, V_k is an $(\varepsilon', \beta, k, m, m', R')$ -superregular equalised partition of $\Gamma \setminus H$.
- (d) There exists a decomposition $\mathcal{D} \cup \mathcal{D}'$ of H where \mathcal{D} is a set of at most βn cycles and \mathcal{D}' is a set of at most β^{-2} edges.

Proof. This can be proved in a similar way as Lemma 4.16, so we only provide a sketch of the proof. Let $\mathcal{D} := \emptyset$. Let C_1, \dots, C_c be the connected components of R' , where, by (ii), $c \leq \beta^{-1}$. For each $x \in V_0$, by (iii) and (iv), $|N_G(x) \setminus V_0| \geq (\alpha - \varepsilon)n$ and thus there exists $i \in [c]$ such that $|N_G(x) \cap V_\Gamma(C_i)| \geq \beta^2 n$. Colour each $x \in V_0$ with such a colour $i \in [c]$.

Apply Theorem 1.1 to decompose $G[V_0]$ into ℓ paths P_1, \dots, P_ℓ and ℓ^* cycles. Add the ℓ^* cycles to \mathcal{D} and note that $|\mathcal{D}|, \ell \leq \varepsilon n$. We apply the arguments of the proof of Lemma 4.16 with $\mathcal{P}_i = \{P_i\}$ for each $i \in [\ell]$. The only difference is that we now have paths with endpoints in V_0 . We adapt to this setting as follows.

Partition the paths P_1, \dots, P_ℓ into monochromatic subpaths and bichromatic edges as in Lemma 4.16. Let $1 \leq i < i' \leq c$. We may assume that the set $\mathcal{Q}_{ii'}$ of bichromatic edges coloured with $\{i, i'\}$ is a matching. Indeed, if $\mathcal{Q}_{ii'}$ contains distinct edges e and e' with a common endpoint, then we can delete e and e' from $\mathcal{Q}_{ii'}$ and consider $e \cup e'$ as a monochromatic path instead. Since for each $1 \leq i < i' \leq c$, we have $|\mathcal{Q}_{ii'}| \leq \ell$, we obtain, in total, at most $\binom{c}{2} \frac{\ell}{2} \leq \sqrt{\varepsilon} n$ additional monochromatic paths.

For each $i \in [c]$, extend each monochromatic path coloured i to a path with internal vertices in V_0 and endpoints in $V_\Gamma(C_i)$. Similarly, for each $1 \leq i < i' \leq c$, extend $\mathcal{Q}_{ii'}$ to a set of vertex-disjoint paths of length 3 with internal vertices in V_0 , an endpoint in $V_\Gamma(C_i)$ and an endpoint in $V_\Gamma(C_{i'})$. Then, one can easily show that we can proceed as in the proof of Lemma 4.16.

One can easily verify that, in the end, we have covered all but at most β^{-2} edges of $G[V_0]$ with at most βn cycles, as desired. \square

5.3. Main step of the decomposition. This section corresponds to Step 3 of the proof overview. Lemma 5.3 will be used to obtain a cycle decomposition in the proof of Theorem 1.10(ii) and Lemma 5.4 will be used to obtain a path decomposition in the proof of Theorem 1.10(i). Lemma 5.5 will play a similar role in the proof of Theorem 1.11.

Lemma 5.3. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \varepsilon \ll \zeta \ll d \ll \beta \leq 1$. Let G, Γ, Γ' be edge-disjoint graphs on the same vertex set V of order n . Assume V_0, V_1, \dots, V_k is a partition of V such that the following hold.*

- (i) V_0, V_1, \dots, V_k is
 - an $(\varepsilon, \geq d, k, m, m', R)$ -superregular equalised partition of G ,
 - an $(\varepsilon, \beta, k, m, m', R')$ -superregular equalised partition of Γ ,
 - an $(\varepsilon, \zeta, k, m, m', R'')$ -superregular equalised partition of Γ' .
- (ii) R' and R'' are edge-disjoint and $R' \cup R'' = R$.
- (iii) G, Γ , and Γ' have the same support clusters.
- (iv) V_0 is a set of isolated vertices in Γ and Γ' . Moreover, $G[V_0]$ is empty.
- (v) Any $x \in V \setminus V_0$ belongs to at least βk superregular pairs of Γ .
- (vi) For any $x \in V_0$, $d_G(x)$ is even.

Then, $G \cup \Gamma \cup \Gamma'$ can be decomposed into edge-disjoint graphs $G', \tilde{\Gamma}$, and H such that $G, \Gamma' \subseteq G' \cup H$, $\tilde{\Gamma} \subseteq \Gamma$, and the following hold.

- (a) $\Delta(H) \leq 13\zeta n$ and V_0 is a set of isolated vertices in H .
- (b) V_0, V_1, \dots, V_k is a $(\zeta, \beta, k, m, m', R')$ -superregular equalised partition of $\tilde{\Gamma}$.
- (c) There exists a decomposition $\mathcal{D} \cup \mathcal{D}_{\text{exc}}$ of G' such that \mathcal{D} is a set of at most $\frac{n}{2} + 2\beta n$ cycles and \mathcal{D}_{exc} is a set of at most β^{-2} exceptional edges.

H should be thought of as a sparse “leftover” and Γ is a graph which we want to use as little as possible. An analogous result can be obtained for path decompositions.

Lemma 5.4. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \varepsilon \ll \zeta \ll d \ll \beta \leq 1$. Let G, Γ , and Γ' be edge-disjoint graphs on the same vertex set V of order n . Assume V_0, V_1, \dots, V_k is a partition of V such that properties (i)–(vi) of Lemma 5.3 hold. Let $U \subseteq V \setminus V_0$ have even size. Then, $G \cup \Gamma \cup \Gamma'$ can be decomposed into edge-disjoint graphs $G', \tilde{\Gamma}$, and H such that $G, \Gamma' \subseteq G' \cup H$, $\tilde{\Gamma} \subseteq \Gamma$, and properties (a) and (b) of Lemma 5.3 are satisfied. Moreover,*

- (c') *there exists a path decomposition \mathcal{D} of G' such that $|\mathcal{D}| \leq \frac{n}{2} + 4\beta n$ and, for any $x \in V \setminus V_0$, \mathcal{D} contains an odd number of paths with x as an endpoint if and only if $x \in U$.*

The next lemma shows that stronger results can be obtained if the reduced graph R of G is assumed to be connected.

Lemma 5.5. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll \varepsilon \ll \zeta \ll d \ll \beta \leq 1$. Let G, Γ , and Γ' be edge-disjoint graphs on the same vertex set V of order n . Assume V_0, V_1, \dots, V_k is a partition of V such that properties (i)–(vi) of Lemma 5.3 hold. Suppose furthermore that R is connected. Then, the following hold.*

- (a) $G \cup \Gamma \cup \Gamma'$ can be decomposed into edge-disjoint graphs $G', \tilde{\Gamma}$, and H such that $G, \Gamma' \subseteq G' \cup H$, $\tilde{\Gamma} \subseteq \Gamma$, and properties (a) and (b) of Lemma 5.3 hold. Furthermore, G' can be decomposed into at most $\frac{\Delta(G)}{2} + 7\zeta n$ cycles.
- (b) Let $U \subseteq V \setminus V_0$ have even size. Then $G \cup \Gamma \cup \Gamma'$ can be decomposed into edge-disjoint graphs $G', \tilde{\Gamma}$, and H such that $G, \Gamma' \subseteq G' \cup H$, $\tilde{\Gamma} \subseteq \Gamma$, and properties (a) and (b) of Lemma 5.4 hold. Furthermore, there exists a path decomposition \mathcal{D} of G' such that $|\mathcal{D}| \leq \max \left\{ \frac{\Delta(G)}{2}, \frac{|U|}{2} \right\} + 8\zeta n$ and each vertex $x \in V \setminus V_0$ is an endpoint of an odd number of paths in \mathcal{D} if and only if $x \in U$.

Lemmas 5.3–5.5 will be proved simultaneously. To obtain a path decomposition, the idea is to insert suitable fictive edges and then construct a cycle decomposition such that each cycle in the decomposition contains exactly one fictive edge.

We will need the following result of [19].

Theorem 5.6 ([19]). *Let $0 < \frac{1}{m} \ll \varepsilon \ll d < 1$ and assume G is a bipartite graph on vertex classes A, B of size m . If G is $[\varepsilon, d]$ -superregular then G contains $\frac{(d-19\varepsilon)m}{2}$ edge-disjoint Hamilton cycles.*

Proof of Lemmas 5.3–5.5. Define

$$\begin{aligned} \varepsilon_1 &:= \varepsilon^{\frac{1}{12}}, & \varepsilon_2 &:= \varepsilon_1^{\frac{1}{7}}, & \varepsilon_3 &:= \varepsilon_2^{\frac{1}{5}}, & \varepsilon_4 &:= \varepsilon_2^{\frac{1}{15}}, & \varepsilon_5 &:= \varepsilon_2^{\frac{1}{75}}, & \varepsilon_6 &:= \varepsilon_2^{\frac{1}{225}}, \\ \varepsilon_7 &:= \varepsilon_2^{\frac{1}{1125}}, & \varepsilon_8 &:= \varepsilon_2^{\frac{1}{3375}}, & \varepsilon_9 &:= \varepsilon_8^{\frac{1}{3375}}, & \zeta_1 &:= \sqrt{2\zeta}, & \zeta_2 &:= \sqrt{\zeta_1}. \end{aligned}$$

Let $i, j \in [k]$ be distinct. Denote by V_{ij} and V_{ji} the support clusters of $G[V_i, V_j]$. If $G[V_i, V_j]$ is empty let $d_{ij} := 0$. Otherwise, by (SRP4), (SRP5), and definition of the reduced graph, $G[V_{ij}, V_{ji}]$ is $[\varepsilon, \geq d]$ -superregular and so we can let d_{ij} be a constant such that $G[V_{ij}, V_{ji}]$ is $[\varepsilon, d_{ij}]$ -superregular. We let G' and H be empty graphs on V . Throughout this proof, we will repeatedly add edges to G' and H , and, whenever we do so, these edges are deleted from $G \cup \Gamma \cup \Gamma'$.

Let $G_{\text{exc}} := \emptyset$. For each connected component C of R and $x \in V_0$, if $|N_G(x) \cap V_G(C)|$ is odd, add exactly one edge of $G[\{x\}, V_G(C)]$ to G_{exc} . Delete the edges in G_{exc} from G . Observe that we may now assume that for each connected component C of R , any $x \in V_0$ has even degree in $G[V_0 \cup V_G(C)]$. The graph G_{exc} will be covered in Step 8. Observe that, by (vi), G_{exc} is empty in the proof of Lemma 5.5.

We now assume R is connected. If it is disconnected, we will apply Steps 1–7 to each connected component of R separately and then cover the potentially remaining edges (i.e. the edges of G_{exc}) in Step 8. Fix $\Delta := \Delta(G)$, $\Delta_0 := \max\{d_G(x) \mid x \in V_0\}$, and $\Delta' := \max\{d_G(x) \mid x \notin V_0\}$. In particular, in what follows, Δ, Δ' , and Δ_0 are left unchanged when we delete some edges from G .

Step 1: Partitioning the edges of G and constructing reservoirs of vertices. We will partition each superregular pair of G into subgraphs of small comparable density. Each subgraph will be assigned a reservoir, that is a small number of vertices that will be set aside to tie paths together later on. To do so, we will partition each cluster into small subclusters of equal size and, in each subgraph, one of these subclusters will play the role of the reservoir.

Let $r := \lfloor \zeta^{-1} \rfloor$ (r will be the number of reservoirs). For each $ij \in E(R)$, let $\ell_{ij} := \lfloor \zeta^{-1} d_{ij} \rfloor$ and apply Lemma 4.8 to partition $G[V_{ij}, V_{ji}]$ into $r\ell_{ij}$ spanning edge-disjoint $[\varepsilon_1, \zeta^2]$ -superregular graphs $G_{1,1}^{ij}, \dots, G_{1,\ell_{ij}}^{ij}, G_{2,1}^{ij}, \dots, G_{r,\ell_{ij}}^{ij}$ and a leftover graph G_0^{ij} which we add to H . Note that,

for each $ij \in E(R)$, $\Delta(G_0^{ij}) \leq (d_{ij} + \varepsilon)m - (\zeta^2 - \varepsilon)m r \ell_{ij} \leq 3\zeta m$ and thus

$$(5.1) \quad \Delta(H) \leq 3\zeta n.$$

For each $i \in [k]$, randomly partition V_i into r subclusters V_i^1, \dots, V_i^r of size ζm . (If $\zeta m \notin \mathbb{N}^*$, then the subclusters will only have sizes roughly ζm , but this does not affect the argument below.) Let $V^\ell := \bigcup_{i \in [k]} V_i^\ell$. Also define $U^\ell := V^\ell \cap U$ for the proof of Lemmas 5.4 and 5.5(b). We claim that the following hold with positive probability.

- (1) For any $ij \in E(R)$, $\ell \in [r]$, and $\ell' \in [\ell_{ij}]$,
 - (1a) $G_{\ell, \ell'}^{ij}[X, Y]$ is $[\varepsilon_2, \zeta^2]$ -superregular for each $X \in \{V_{ij} \setminus V_i^\ell, V_{ij} \cap V_i^\ell\}$ and $Y \in \{V_{ji} \setminus V_j^\ell, V_{ji} \cap V_j^\ell\}$;
 - (1b) if $ij \in E(R')$, then $\Gamma_{ij\ell} := \Gamma[V_{ij} \cap V_i^\ell, V_{ji} \cap V_j^\ell]$ is $[\varepsilon_2, \beta]$ -superregular;
 - (1c) if $ij \in E(R'')$, then $\Gamma'_{ij\ell} := \Gamma'[V_{ij} \cap V_i^\ell, V_{ji} \cap V_j^\ell]$ is $[\varepsilon_2, \zeta]$ -superregular.
- (2) For any $ij \in E(R)$ and $\ell \in [r]$, $|V_{ij} \cap V_i^\ell| = (1 \pm \varepsilon_1)\zeta m'$.
- (3) For each $\ell \in [r]$ and $x \in V_0$, $|N_G(x) \cap V^\ell| = \frac{|N_G(x)|}{r} \pm \varepsilon n$.

Additionally, for the proof of Lemmas 5.4 and 5.5(b), the following holds with positive probability.

- (4) For each $\ell \in [r]$, $|U^\ell| = \frac{|U|}{r} \pm \varepsilon n$.

Indeed, by Lemma 4.10, (1) and (2) hold with high probability, and, a simple application of Lemma 3.1 shows that (3) and (4) hold with high probability. Therefore, by a union bound, we may assume that (1)–(4) are satisfied.

For each $ij \in E(R)$, $\ell \in [r]$, and $\ell' \in [\ell_{ij}]$, define $\tilde{G}_{\ell, \ell'}^{ij} := G_{\ell, \ell'}^{ij}[V_{ij} \setminus V_i^\ell, V_{ji} \setminus V_j^\ell]$ and add all edges of $G_{\ell, \ell'}^{ij}[V_{ij} \cap V_i^\ell, V_{ji} \cap V_j^\ell]$ to H . By (1a), we add, in total, at most $(\zeta^2 + \varepsilon_2)\zeta m \cdot \zeta^{-1} \cdot k \leq 2\zeta^2 n$ edges incident to each vertex, so, by (5.1),

$$(5.2) \quad \Delta(H) \leq 4\zeta n.$$

Step 2: Equalising the support cluster sizes. For any distinct $i, j \in [k]$ and $\ell \in [r]$ in turn, we now construct a subset $V_{ij\ell} \subseteq V_{ij} \setminus V_i^\ell$ of size $m'' := (1 - \zeta - \varepsilon)m'$ by removing exactly $|V_{ij} \setminus V_i^\ell| - m''$ vertices. We build these sets one by one and, in each step, we only remove vertices which have already been removed fewer than $\sqrt{\varepsilon_1}k$ times in the construction so far. This is possible since in each step, by (2), we need to remove at most $2\varepsilon_1 m$ vertices, and so, in each step, there are at most ζm vertices which we are not allowed to remove anymore. On the other hand, by (2), $|V_{ij} \setminus V_i^\ell| \geq (1 - 2\zeta)m$ for any distinct $i, j \in [k]$, and $\ell \in [r]$

For any $ij \in E(R)$, $\ell \in [r]$, and $\ell' \in [\ell_{ij}]$, define $\hat{G}_{\ell, \ell'}^{ij} := \tilde{G}_{\ell, \ell'}^{ij}[V_{ij\ell}, V_{ji\ell}]$. By (1a) and Lemma 4.2, $\hat{G}_{\ell, \ell'}^{ij}$ is $[2\sqrt{\varepsilon_2}, \zeta^2]$ -superregular. Add to H all edges of $\tilde{G}_{\ell, \ell'}^{ij} \setminus \hat{G}_{\ell, \ell'}^{ij}$. Since $\tilde{G}_{\ell, \ell'}^{ij}$ is obtained from $G_{\ell, \ell'}^{ij}$ by deleting at most $2\varepsilon_1 m$ vertices from each cluster, for each $x \in V_{ij\ell}$, x has degree at most $2\varepsilon_1 m$ in $\tilde{G}_{\ell, \ell'}^{ij} \setminus \hat{G}_{\ell, \ell'}^{ij}$. Moreover, for each $i \in [k]$ and $x \in V_i$, there are at most $\sqrt{\varepsilon_1}k$ pairs $(j, \ell) \in [k] \times [r]$ such that $x \in V(\tilde{G}_{\ell, \ell'}^{ij}) \setminus V(\hat{G}_{\ell, \ell'}^{ij})$. Thus, we have added to H at most $2\sqrt{\varepsilon_1}n$ edges incident to each vertex and, thus, by (5.2),

$$(5.3) \quad \Delta(H) \leq 5\zeta n.$$

Step 3: Decomposing non-exceptional edges of G into long paths with endpoints in reservoirs. For each $ij \in E(R)$, $\ell \in [r]$, and $\ell' \in [\ell_{ij}]$, apply Theorem 5.6 with $\hat{G}_{\ell, \ell'}^{ij}$, ζ^2 , and $2\sqrt{\varepsilon_2}$ playing the roles of G , d , and ε to obtain a set $\mathcal{H}_{\ell, \ell'}^{ij}$ of $h := \frac{(\zeta^2 - 38\sqrt{\varepsilon_2})m''}{2}$ edge-disjoint Hamilton cycles of $\hat{G}_{\ell, \ell'}^{ij}$. We turn each cycle in $\mathcal{H}_{\ell, \ell'}^{ij}$ into a path one by one by deleting an edge xy such that no edge incident to x or y has already been deleted from $\mathcal{H}_{\ell, \ell'}^{ij}$. This is possible since $|\mathcal{H}_{\ell, \ell'}^{ij}| \leq \frac{\zeta^2 m''}{2}$ and each cycle in $\mathcal{H}_{\ell, \ell'}^{ij}$ is of length $2m''$. We add all these edges as well as

all the edges in $E(\widehat{G}_{\ell,\ell'}^{ij}) \setminus E(\mathcal{H}_{\ell,\ell'}^{ij})$ to H . Thus, we add at most $(1 + 40\sqrt{\varepsilon_2}m'')r\zeta^{-1}k \leq \zeta^2n$ edges incident to each vertex, and so, by (5.3),

$$(5.4) \quad \Delta(H) \leq 6\zeta n.$$

We now extend the paths in $\mathcal{H}_{\ell,\ell'}^{ij}$ to paths with internal vertices in $V_{ij\ell} \cup V_{ji\ell}$ and endpoints in $V_i^\ell \cup V_j^\ell$ one by one as follows. Given an (x, y) -path P in $\mathcal{H}_{\ell,\ell'}^{ij}$ with $x \in V_{ij\ell}$ and $y \in V_{ji\ell}$, pick $x' \in V_j^\ell$ and $y' \in V_i^\ell$ such that $xx', yy' \in E(G_{\ell,\ell'}^{ij})$ and $\mathcal{H}_{\ell,\ell'}^{ij}$ contains fewer than ε^2m paths with x' as endpoint and similarly for y' . Replace P in $\mathcal{H}_{\ell,\ell'}^{ij}$ by the path $x'xPy y'$ and delete xx', yy' from $G_{\ell,\ell'}^{ij}$. Note that the existence of x' and y' is guaranteed by (1a) and (2), and the fact that $|\mathcal{H}_{\ell,\ell'}^{ij}| \leq \frac{\zeta^2m}{2}$.

Once all paths in $\mathcal{H}_{\ell,\ell'}^{ij}$ have been extended as above, add all remaining edges of $G_{\ell,\ell'}^{ij}[V_{ij} \cap V_i^\ell, V_{ji} \setminus V_j^\ell]$ and $G_{\ell,\ell'}^{ij}[V_{ij} \setminus V_i^\ell, V_{ji} \cap V_j^\ell]$ to H . Note that by (1a), if $x \in V_i^\ell$, then $G_{\ell,\ell'}^{ij}[V_{ij} \cap V_i^\ell, V_{ji} \setminus V_j^\ell]$ contains at most $2\zeta^2m$ edges incident to x , and if $x \in V_i \setminus V_i^\ell$, $G_{\ell,\ell'}^{ij}[V_{ij} \setminus V_i^\ell, V_{ji} \cap V_j^\ell]$ contains at most $2\zeta^3m$ edges incident to x . This holds for any $ij \in E(R)$, $\ell \in [r]$, and $\ell' \in [\ell_{ij}]$ so, in total, we have added to H at most $4\zeta n$ edges incident to each vertex and thus, by (5.4)

$$(5.5) \quad \Delta(H) \leq 10\zeta n.$$

Moreover, for each $ij \in E(R)$, $\ell \in [r]$, and $\ell' \in [\ell_{ij}]$, add all edges of $\mathcal{H}_{\ell,\ell'}^{ij}$ to G' . Note that all (remaining) edges of G now have exactly one endpoint in V_0 .

Step 4: Combining the paths into sets of vertex-disjoint paths. Let \widehat{R} be the multigraph obtained from R by replacing each edge $ij \in E(R)$ by ℓ_{ij} parallel edges denoted $e_{ij}^1, \dots, e_{ij}^{\ell_{ij}}$.

Claim 1. $\Delta(\widehat{R}) \leq \frac{\Delta'}{\zeta m} + \sqrt{\varepsilon}k$.

Proof of Claim. Let $i \in [k]$ and recall that the graphs G , G' , and H at the end of Step 1 form a decomposition of the original graph G . Clearly, we have

$$e_{G \cup G' \cup H}(V_i, V \setminus V_0) \leq \Delta' m.$$

Moreover, by (i),

$$e_{G \cup G' \cup H}(V_i, V \setminus V_0) \geq \sum_{j \neq i} (d_{ij} - \varepsilon)(1 - \varepsilon)^2 m^2 \geq \sum_{j \neq i} (\zeta \ell_{ij} - 4\varepsilon) m^2 \geq \zeta d_{\widehat{R}}(V_i) m^2 - 4\varepsilon m^2 k$$

and the claim holds. \diamond

Note that there are at most ζ^{-1} parallel edges between any two vertices of \widehat{R} . Thus, we can apply Claim 1 and Theorem 4.17 to fix a decomposition $\mathcal{D}_{\widehat{R}}$ of \widehat{R} into at most $\frac{\Delta'}{\zeta m} + 2\sqrt{\varepsilon}k$ matchings. For each $M \in \mathcal{D}_{\widehat{R}}$ and each $\ell \in [r]$, we decompose $\bigcup_{e_{ij}^{\ell'} \in M} \mathcal{H}_{\ell,\ell'}^{ij}$ into h disjoint sets of paths containing exactly one path of $\mathcal{H}_{\ell,\ell'}^{ij}$ for each $e_{ij}^{\ell'} \in M$.

Let \mathcal{P} be the collection of all the $|\mathcal{D}_{\widehat{R}}|rh$ linear forests obtained. Note that for each $x \in V \setminus V_0$, all non-exceptional edges incident to x are covered by paths in $\bigcup \mathcal{P}$, apart from those lying in H . Thus, $\frac{\Delta'}{2} - 6\zeta n \leq |\mathcal{P}| \leq rh \left(\frac{\Delta'}{\zeta m} + 2\sqrt{\varepsilon}k \right) \leq \frac{\Delta'}{2} + \sqrt{\varepsilon}n$. We also note that for each $\mathcal{P} \in \mathcal{P}$ there exists $r(\mathcal{P}) \in [r]$ such that \mathcal{P} is a set of vertex-disjoint paths with endpoints in $V^{r(\mathcal{P})}$ and internal vertices in $V \setminus V^{r(\mathcal{P})}$. For each $\ell \in [r]$, let $\mathcal{P}_\ell := \{\mathcal{P} \in \mathcal{P} \mid r(\mathcal{P}) = \ell\}$. By construction, $|\mathcal{P}_1| = \dots = |\mathcal{P}_r| = \frac{|\mathcal{P}|}{r}$.

Step 5: Including exceptional edges. For each $\ell \in [r]$, we add exceptional edges to the linear forests in \mathcal{P}_ℓ as follows. If possible, pick $\mathcal{P} \in \mathcal{P}_\ell$ such that we have not yet added exceptional edges to \mathcal{P} and such that G contains a set of paths \mathcal{P}_{exc} satisfying the following.

- (I) \mathcal{P}_{exc} is a set of vertex-disjoint paths of G of length of 2.

- (II) The paths in \mathcal{P}_{exc} have their endpoints in $V^\ell \setminus V(\mathcal{P})$ and internal vertex in V_0 .
- (III) $V(\mathcal{P}_{\text{exc}}) \cap V_0$ is the set of vertices $x \in V_0$ such that $|N_G(x) \cap V^\ell|$ is maximum.
- (IV) $|V(\mathcal{P}_{\text{exc}}) \cap V_i^\ell| \leq \zeta^2 m$ for each $i \in [k]$.

Fix such a set \mathcal{P}_{exc} and add the paths in \mathcal{P}_{exc} to \mathcal{P} . Add the edges of \mathcal{P}_{exc} to G' . We repeat this procedure until there is no such \mathcal{P} . Then, we claim that the following holds.

Claim 2. For each $x \in V_0$, $d_G(x) \leq \max\{\Delta_0 - \Delta' + 13\zeta n, \zeta n\}$.

Proof of Claim. Note that it is enough to show that for each $\ell \in [r]$ and $x \in V_0$, we have $|N_G(x) \cap V^\ell| \leq \max\{\frac{1}{r}(\Delta_0 - \Delta' + 13\zeta n), \sqrt{\varepsilon}n\}$.

Let $\ell \in [r]$. Suppose first that we have added a set \mathcal{P}_{exc} satisfying (I)–(IV) to each $\mathcal{P} \in \mathcal{P}_\ell$. By (3), each vertex in V_0 initially had at most $\frac{\Delta_0}{r} + \varepsilon n$ neighbours in V^ℓ . Thus, by (II) and (III), we now have $|N_G(x) \cap V^\ell| \leq \frac{\Delta_0}{r} + \varepsilon n - 2|\mathcal{P}_\ell| \leq \frac{\Delta_0}{r} + \varepsilon n - \frac{2}{r}(\frac{\Delta'}{2} - 6\zeta n) \leq \frac{1}{r}(\Delta_0 - \Delta' + 13\zeta n)$ for each $x \in V_0$.

Suppose now that there exists $\mathcal{P} \in \mathcal{P}_\ell$ which does not contain any exceptional edges. We claim that for any $x \in V_0$, we have $|N_G(x) \cap V^\ell| \leq \sqrt{\varepsilon}n$. Suppose not. Let x_1, \dots, x_s be an enumeration of the vertices $x \in V_0$ such that $|N_G(x) \cap V^\ell|$ is maximum. By assumption, we have $|N_G(x_i) \cap V^\ell| > \sqrt{\varepsilon}n$. Suppose inductively that, for some $0 \leq i \leq s$, we have constructed a set of paths $\mathcal{P}_{\text{exc}}^i$ satisfying (I), (II), and (IV) and such that $|V(\mathcal{P}_{\text{exc}}^i) \cap V_0| = \{x_j \mid j \in [i]\}$. If $i < s$, we construct $\mathcal{P}_{\text{exc}}^{i+1}$ as follows. Let X be the set of indices $j \in [\ell]$ such that $|V(\mathcal{P}_{\text{exc}}^i) \cap V_j^\ell| = \zeta^2 m$. Let $Y := (N_G(x_{i+1}) \cap V_\ell) \setminus (V(\mathcal{P}) \cup V(\mathcal{P}_{\text{exc}}^i) \cup \bigcup_{j \in X} V_j^\ell)$. Then, $|Y| \geq \sqrt{\varepsilon}n - k - 2i - \frac{2i}{\zeta^2 m} \cdot \zeta m \geq 2$. Let $y, y' \in Y$ be distinct. Set $\mathcal{P}_{\text{exc}}^{i+1} := \mathcal{P}_{\text{exc}}^i \cup \{yx_{i+1}y'\}$. By construction, $\mathcal{P}_{\text{exc}}^{i+1}$ satisfies (I), (II), and (IV) and $|V(\mathcal{P}_{\text{exc}}^{i+1}) \cap V_0| = \{x_j \mid j \in [i+1]\}$. Therefore, we can construct a set $\mathcal{P}_{\text{exc}}^s$ satisfying (I)–(IV), a contradiction. Thus, $|N_G(x) \cap V^\ell| \leq \sqrt{\varepsilon}n$ for all $x \in V_0$. \diamond

We now split most of the remaining exceptional edges into sets of vertex-disjoint paths, in a similar way as above. Let $\mathcal{P}' := \emptyset$. Assume there exists a set of paths \mathcal{P}_{exc} satisfying (I) and the following.

- (II') The paths in \mathcal{P}_{exc} have their endpoints in $V \setminus V_0$ and internal vertex in V_0 .
- (III') $V(\mathcal{P}_{\text{exc}}) \cap V_0$ is the set of vertices $x \in V_0$ such that $|N_G(x) \cap V \setminus V_0|$ is maximum.
- (IV') $|V(\mathcal{P}_{\text{exc}}) \cap V_i| \leq \zeta m$ for each $i \in [k]$.

Then add such a set \mathcal{P}_{exc} to \mathcal{P}' and add the edges of \mathcal{P}_{exc} to G' . We repeat this procedure until we cannot find any \mathcal{P}_{exc} as above. Then, one can show using similar arguments as in the proof of Claim 2 that $d_G(x) \leq \sqrt{\varepsilon}n$ for each $x \in V_0$.

We will now form at most $4\sqrt{\varepsilon}n$ sets of linear forests which cover all remaining exceptional edges of G . Assume that $\mathcal{P}_1, \dots, \mathcal{P}_{4\sqrt{\varepsilon}n}$ are (possibly empty) edge-disjoint sets of paths such that for each $i \in [4\sqrt{\varepsilon}n]$ the following are satisfied and such that $\sum_{i \in [4\sqrt{\varepsilon}n]} |\mathcal{P}_i|$ is maximal.

- (I'') \mathcal{P}_i is a set of vertex-disjoint paths of G of length 2.
- (II'') The paths in \mathcal{P}_i have their endpoints in $V \setminus V_0$ and internal vertex in V_0 .
- (IV'') $|V(\mathcal{P}_i) \cap V_j| \leq \zeta m$ for each $j \in [k]$.

Add all edges of $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_{4\sqrt{\varepsilon}n}$ to G' . If $d_G(x) = 0$ for all $x \in V_0$, add $\mathcal{P}_1, \dots, \mathcal{P}_{4\sqrt{\varepsilon}n}$ to \mathcal{P}' and we are done. We may therefore assume that there is $x \in V_0$ with $d_G(x) \geq 2$. Pick distinct $y, z \in N_G(x)$ and let $i, i' \in [k]$ be such that $y \in V_i$ and $z \in V_{i'}$. By maximality, we only need to find $j \in [4\sqrt{\varepsilon}n]$ such that (I''), (II''), and (IV'') are still satisfied if we add yxz to \mathcal{P}_j and thus obtain a contradiction. By construction, x belongs to fewer than $\sqrt{\varepsilon}n$ of the \mathcal{P}_j , and, since $|V_0| \leq \varepsilon n$, each of y and z belong to fewer than εn of the \mathcal{P}_j . Moreover, there are at most $\frac{\varepsilon n m}{\zeta m - 2} \leq \sqrt{\varepsilon}n$ indices $j \in [4\sqrt{\varepsilon}n]$ such that $|V(\mathcal{P}_j) \cap V_i| > \zeta m - 2$ and similarly, there are at most $\sqrt{\varepsilon}n$ indices $j \in [4\sqrt{\varepsilon}n]$ such that $|V(\mathcal{P}_j) \cap V_{i'}| > \zeta m - 2$. Thus, there are at least $4\sqrt{\varepsilon}n - \sqrt{\varepsilon}n - 2\varepsilon n - 2\sqrt{\varepsilon}n > 0$ indices j such that we can add the path yxz to \mathcal{P}_j and (I''), (II''), and (IV'') are still satisfied.

To summarise, we have constructed sets \mathcal{P}_ℓ , for each $\ell \in [r]$, and a set \mathcal{P}' satisfying the following.

- (A) For each $\ell \in [r]$ and $\mathcal{P} \in \mathcal{P}_\ell$, \mathcal{P} is a set of vertex-disjoint paths with endpoints in V^ℓ and internal vertices in $V \setminus V_\ell$. Moreover, $|\mathcal{P}| \leq \frac{k}{2} + \varepsilon n$ and $|V(\mathcal{P}) \cap V_i^\ell| \leq k + \zeta^2 m$ for each $i \in [k]$.
- (B) For each $\mathcal{P} \in \mathcal{P}'$, \mathcal{P} is a set of vertex-disjoint paths with endpoints in $V \setminus V_0$ and internal vertices in V_0 . Moreover, $|\mathcal{P}| \leq \varepsilon n$ and $|V(\mathcal{P}) \cap V_i| \leq \zeta m$ for each $i \in [k]$.
- (C) For each $x \in V \setminus V_0$, there are at most $2\varepsilon n$ paths in $\bigcup(\mathcal{P} \cup \mathcal{P}')$ which have x as an endpoint (recall $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_r$).
- (D) By Step 4, Claim 2, and the above construction, $|\mathcal{P} \cup \mathcal{P}'| \leq \frac{\Delta}{2} + 7\zeta n$. Moreover, G is now empty.

Indeed, in order to check (C), recall from Step 3 that, for each $x \in V_i^\ell$, each $\mathcal{H}_{\ell,\ell}^{ij}$ contains at most $\varepsilon^2 m$ paths with x as an endpoint.

Step 6: Including fictive edges. We ignore this step for the proof of Lemmas 5.3 and 5.5(a). For the proof of Lemmas 5.4 and 5.5(b), we construct a multiset E_{fict} of fictive edges on $V \setminus V_0$. As discussed in Section 4.4, we view edges in E_{fict} as distinct from each other and from edges in G, G', Γ, Γ' , and H . We will add a fictive edge to each linear forest in $\mathcal{P} \cup \mathcal{P}'$. Moreover, in order to satisfy (c'), we make sure that for any $x \in V \setminus V_0$, E_{fict} contains an odd number of edges incident to x (counting multiplicity) if and only if $x \in U$.

Start with $E_{\text{fict}} = \emptyset$. In what follows, we denote by U_{even} the set of vertices in U which are incident to an even number of edges in E_{fict} and, for each $\ell \in [r]$, we denote by U_{even}^ℓ the set $U_{\text{even}} \cap U^\ell$. In what follows, we will update U_{even} and U_{even}^ℓ at each step of our algorithm. For each $\ell \in [r]$, we add a fictive edge to each linear forest in \mathcal{P}_ℓ as follows. Assume $\mathcal{P} \in \mathcal{P}_\ell$ does not contain a fictive edge yet. If there exist distinct $x, y \in U_{\text{even}}^\ell \setminus V(\mathcal{P})$, add the edge xy to E_{fict} and to \mathcal{P} . If there are no such x and y , then note that, by (A), $|U_{\text{even}}^\ell| \leq |V(\mathcal{P}) \cap V^\ell| + 1 \leq 2|\mathcal{P}| + 1 \leq 3\varepsilon n$ and proceed as follows. If \mathcal{P} is the only linear forest in \mathcal{P}_ℓ which does not contain a fictive edge, we remove \mathcal{P} from \mathcal{P}_ℓ , add all its edges to H and we are done. We note that this increases the maximum degree of H by at most 2. Otherwise, pick $\mathcal{P}' \in \mathcal{P}_\ell \setminus \{\mathcal{P}\}$ such that \mathcal{P}' does not contain a fictive edge. Note that, by (A), $|V^\ell \cap V(\mathcal{P} \cup \mathcal{P}')| \leq \sqrt{\varepsilon} n$. Moreover, by (D), there are at most $\frac{2|\mathcal{P}_\ell|}{\varepsilon n} \leq 2\varepsilon^{-1}$ vertices in V_ℓ which are incident to at least εn edges in E_{fict} . Thus, we can choose distinct $x, y \in V^\ell \setminus V(\mathcal{P} \cup \mathcal{P}')$ such that E_{fict} contains fewer than εn edges incident to x and fewer than εn edges incident to y . Add the edge xy to both \mathcal{P} and \mathcal{P}' , and, add two edges between x and y to E_{fict} . We repeat this procedure until each linear forest in \mathcal{P}_ℓ contains a fictive edge.

We then proceed similarly to add a fictive edge to each linear forest in \mathcal{P}' , now using (B) instead of (A) and allowing fictive edges to have endpoints in $V \setminus V_0$ instead of V^ℓ for some $\ell \in [r]$. Once each linear forest in \mathcal{P}' contains a fictive edge, observe that we have added at most $r + 1$ linear forests to H , so, by (5.5),

$$(5.6) \quad \Delta(H) \leq 11\zeta n.$$

Moreover, the following holds.

Claim 3. *The set U_{even} has even size. Moreover, $|U_{\text{even}}| \leq \max\{|U| - 2(|\mathcal{P} \cup \mathcal{P}'| - (r + 1)), \sqrt{\varepsilon} n\}$.*

Proof of Claim. By construction and since $|U|$ is even, $|U_{\text{even}}|$ is even. For the second part of the claim, we distinguish three cases.

Firstly, assume that for any distinct $\mathcal{P}, \mathcal{P}' \in \mathcal{P} \cup \mathcal{P}'$, the fictive edge of \mathcal{P} is vertex-disjoint from the fictive edge of \mathcal{P}' . Then we clearly have $|U_{\text{even}}| \leq |U| - 2(|\mathcal{P} \cup \mathcal{P}'| - (r + 1))$.

Secondly, assume there exists $\ell \in [r]$ such that there exist distinct $\mathcal{P}, \mathcal{P}' \in \mathcal{P}_\ell$ and $x, y \in V^\ell$ such that both \mathcal{P} and \mathcal{P}' contain a fictive edge between x and y . Then, by construction, $|U_{\text{even}}^\ell| \leq 3\varepsilon n$, and so $|U^\ell| \leq 2|\mathcal{P}_\ell| + 3\varepsilon n$. By (4), for any $\ell' \in [r]$, we have $|U^{\ell'}| \leq 2|\mathcal{P}_{\ell'}| + 5\varepsilon n$ and, thus, $|U_{\text{even}}^{\ell'}| \leq 5\varepsilon n$. Therefore, $|U_{\text{even}}| \leq \sqrt{\varepsilon} n$, as desired.

Thirdly, assume that there exist distinct $\mathcal{P}, \mathcal{P}' \in \mathcal{P}'$ and $x, y \in V \setminus V_0$ such that \mathcal{P} and \mathcal{P}' both contain a fictive edge between x and y . Then by construction $|U_{\text{even}}| \leq 3\epsilon n$. \diamond

Pair all vertices in U_{even} and for each pair (x, y) , add xy to E_{fict} and $\{xy\}$ to \mathcal{P}' . By construction, (4), (A)–(D), and Claim 3, the following hold.

- (A') For each $\ell \in [r]$ and $\mathcal{P} \in \mathcal{P}_\ell$, \mathcal{P} is a set of vertex-disjoint paths with endpoints in V^ℓ and internal vertices in $V \setminus V_\ell$. Moreover, $|\mathcal{P}| \leq \frac{k}{2} + \epsilon n + 1$ and $|V(\mathcal{P}) \cap V_i^\ell| \leq 2\zeta^2 m$ for each $i \in [k]$.
- (B') For each $\mathcal{P} \in \mathcal{P}'$, \mathcal{P} is a set of vertex-disjoint paths with endpoints in $V \setminus V_0$ and internal vertices in V_0 . Moreover, $|\mathcal{P}| \leq \epsilon n + 1$ and $|V(\mathcal{P}) \cap V_i| \leq 2\zeta m$ for each $i \in [k]$.
- (C') For each $x \in V \setminus V_0$, there are at most $4\epsilon n$ paths in $\bigcup(\mathcal{P} \cup \mathcal{P}')$ which have x as an endpoint.
- (D') By (D), Claim 3, and construction, $|\mathcal{P} \cup \mathcal{P}'| \leq \max\left\{\frac{\Delta}{2}, \frac{|U|}{2}\right\} + 8\zeta n$. Moreover, G is now empty.
- (E') Each set in $\mathcal{P} \in \mathcal{P} \cup \mathcal{P}'$ contains exactly one edge of E_{fict} . Moreover, for each $x \in V \setminus V_0$, E_{fict} contains an odd number of edges incident to x if and only if $x \in U$.

Step 7: Tying each set of paths into a cycle. We now tie each linear forest $\mathcal{P} \in \mathcal{P} \cup \mathcal{P}'$ into a cycle using edges of $\Gamma \cup \Gamma'$. This is achieved by successively applying Lemmas 4.12, 4.13, and 4.15 several times as follows.

For each $\ell \in [r]$, we tie the paths of each linear forest in \mathcal{P}_ℓ as follows. Let Γ_ℓ be the graph on vertex set $V_0 \cup V^\ell$ and edge set $\bigcup_{ij \in E(R'_\ell)} E(\Gamma_{ij\ell})$. Note that by (1b), $V_0, V_1^\ell, \dots, V_k^\ell$ is an $(\epsilon_2, \beta, k, \zeta m, R'_\ell)$ -superregular partition of Γ_ℓ , where $V_i^\ell V_j^\ell \in E(R'_\ell)$ if and only if $V_i V_j \in E(R')$. Define Γ'_ℓ similarly. Moreover, by (1c), $V_0, V_1^\ell, \dots, V_k^\ell$ is an $(\epsilon_2, \zeta, k, \zeta m, R''_\ell)$ -superregular partition of Γ'_ℓ , where $V_i^\ell V_j^\ell \in E(R''_\ell)$ if and only if $V_i V_j \in E(R'')$.

For each $\ell \in [r]$, we can then successively apply Lemmas 4.12, 4.13, and 4.15 as follows. Write $\mathcal{P}_\ell = \{\mathcal{P}_1, \dots, \mathcal{P}_{\ell'}\}$. First, we apply Lemma 4.12 with $\Gamma_\ell, \zeta m, V_1^\ell, \dots, V_k^\ell, \epsilon_2, R'_\ell, \ell', \mathcal{P}_1, \dots, \mathcal{P}_{\ell'}$, and 2ζ playing the roles of $\Gamma, m, V_1, \dots, V_k, \epsilon, R, \ell, \mathcal{P}_1, \dots, \mathcal{P}_\ell$, and ζ , respectively. We thus obtain disjoint $E_1, \dots, E_{\ell'} \subseteq E(\Gamma_\ell)$ such that the following hold. For any distinct $i, j \in [k]$, and $x \in V_i^\ell$, the set $E := E_1 \cup \dots \cup E_{\ell'}$ contains at most $\epsilon_3 \zeta m$ edges of $\Gamma_\ell[V_i^\ell, V_j^\ell]$ which are incident to x , and, thus, by Lemma 4.2, it follows that $V_0, V_1^\ell, \dots, V_k^\ell$ is an $(\epsilon_4, \beta, k, \zeta m, R'_\ell)$ -superregular partition of $\Gamma_\ell \setminus E$. Moreover, for any $i \in [\ell']$ and $j \in [k]$, $|V(\mathcal{P}_i \cup E_i) \cap V_j^\ell| \leq \zeta_1 \zeta m$. Finally, for any $i \in [\ell']$, by using each edge in E_i exactly once, we can tie some of the paths in \mathcal{P}_i to form a set of vertex-disjoint paths \mathcal{Q}_i such that, for any $j \in [k]$, at most $2\beta^{-2}$ paths in \mathcal{Q}_i have an endpoint in V_j^ℓ .

We then apply Lemma 4.13 with $\Gamma_\ell \setminus E, \zeta m, V_1^\ell, \dots, V_k^\ell, \epsilon_4, R'_\ell, \ell', \mathcal{Q}_1, \dots, \mathcal{Q}_{\ell'}$, and ζ_1 playing the roles of $\Gamma, m, V_1, \dots, V_k, \epsilon, R, \ell, \mathcal{P}_1, \dots, \mathcal{P}_\ell$, and ζ , respectively. We thus obtain disjoint $E'_1, \dots, E'_{\ell'} \subseteq E(\Gamma_\ell) \setminus E$ satisfying the following. For any distinct $i, j \in [k]$, and $x \in V_i^\ell$, the set $E' := E'_1 \cup \dots \cup E'_{\ell'}$ contains at most $\epsilon_5 \zeta m$ edges of $\Gamma_\ell[V_i^\ell, V_j^\ell]$ which are incident to x , and, thus, by Lemma 4.2, it follows that $V_0, V_1^\ell, \dots, V_k^\ell$ is an $(\epsilon_6, \beta, k, \zeta m, R'_\ell)$ -superregular partition of $\Gamma_\ell \setminus (E \cup E')$. Moreover, for any $i \in [\ell']$ and $j \in [k]$, $|V(\mathcal{Q}_i \cup E'_i) \cap V_j^\ell| \leq \zeta_2 \zeta m$. Finally, for any $i \in [\ell']$, by using each edge in E'_i exactly once, we can tie the paths in \mathcal{Q}_i to form a set of vertex-disjoint paths \mathcal{Q}'_i such that, for any component C of R'_ℓ , \mathcal{Q}'_i contains at most one path with an endpoint in $V_{\Gamma_\ell}(C)$.

We then apply Lemma 4.15 with $\Gamma_\ell \setminus (E \cup E'), \zeta m, V_1^\ell, \dots, V_k^\ell, \epsilon_6, R'_\ell, \ell', \mathcal{Q}'_1, \dots, \mathcal{Q}'_{\ell'}, \zeta_2, \Gamma'_\ell$, and R'_ℓ playing the roles of $\Gamma, m, V_1, \dots, V_k, \epsilon, R, \ell, \mathcal{P}_1, \dots, \mathcal{P}_\ell, \zeta, \Gamma'$, and R' , respectively. We thus obtain disjoint $E''_1, \dots, E''_{\ell'} \subseteq E(\Gamma_\ell \cup \Gamma'_\ell) \setminus (E \cup E')$ satisfying the following. For any distinct $i, j \in [k]$, and $x \in V_i^\ell \cup V_j^\ell$, the set $E'' := E''_1 \cup \dots \cup E''_{\ell'}$ contains at most $\epsilon_7 \zeta m$ edges of $\Gamma_\ell[V_i^\ell, V_j^\ell]$ which are incident to x and at most $\epsilon_7 \zeta m$ edges of $\Gamma'_\ell[V_i^\ell, V_j^\ell]$ which are incident to x . Moreover, $\mathcal{Q}'_i \cup E''_i$ forms a cycle, i.e. $\mathcal{P}_\ell \cup (E \cup E' \cup E'')$ admits a cycle decomposition \mathcal{D}_ℓ of size $|\mathcal{P}_\ell|$. Add all edges in $E \cup E' \cup E''$ to G' . Proceed in this way for each $\ell \in [r]$.

Observe that by construction and since the reservoirs are pairwise disjoint, Lemma 4.2 implies that V_0, V_1, \dots, V_k is now an $(\varepsilon_8, \beta, k, m, m', R')$ -superregular equalised partition of Γ and an $(\varepsilon_8, \zeta, k, m, m', R'')$ -superregular equalised partition of Γ' .

We proceed similarly to obtain a set $E^* \subseteq E(\Gamma \cup \Gamma')$ such that $\mathcal{P}' \cup E^*$ admits a cycle decomposition \mathcal{D}' of size $|\mathcal{P}'|$. Add the edges in E^* to G' . Then, Lemma 4.2 implies that V_0, V_1, \dots, V_k is an $(\varepsilon_9, \beta, k, m, m', R')$ -superregular equalised partition of Γ . Add all remaining edges of Γ' to H . By (i), we add at most $(\zeta + \varepsilon)n$ edges incident to each vertex. Thus, by (5.6), $\Delta(H) \leq 13\zeta n$, as desired for (a).

Let $\mathcal{D} := \bigcup_{\ell \in [r]} \mathcal{D}_\ell \cup \mathcal{D}'$. Observe that for the proof of Lemmas 5.3 and 5.5(a), by (D) and construction, \mathcal{D} is a cycle decomposition of G' of size at most $|\mathcal{P} \cup \mathcal{P}'| \leq \frac{\Delta}{2} + 7\zeta n$. For the proof of Lemmas 5.4 and 5.5(b), remove all fictive edges from \mathcal{D} . Then, by (D') and (E'), \mathcal{D} is now a path decomposition of G' of size at most $\max\left\{\frac{\Delta}{2}, \frac{|U|}{2}\right\} + 8\zeta n$. Moreover, each vertex $x \in V \setminus V_0$ is an endpoint of an odd number of paths in \mathcal{D} if and only if $x \in U$. This completes the proof of Lemma 5.5 (where we set $\tilde{\Gamma} := \Gamma$).

Step 8: Covering the remaining exceptional edges. If R is disconnected, we apply the above argument to each component of R . More precisely, for each connected component C of R , we apply Steps 1–7 with $R[C]$ and $G[V_0 \cup V_G(C)]$ playing the roles of R and G , and, for the proof of Lemma 5.4, $U \cap V_G(C)$ playing the role of U . In particular, observe that for each component C of R , $\Delta(G[V_0 \cup V_G(C)]) \leq |V_G(C)| + \varepsilon n$. Moreover, by (ii) and (v), R has at most β^{-1} components. Therefore, we obtain, in the proof of Lemma 5.3 (Lemma 5.4), a cycle (path) decomposition \mathcal{D} of G of size at most $\frac{n}{2} + \frac{8\zeta n}{\beta} + \frac{\varepsilon n}{\beta} \leq \frac{n}{2} + \beta n$. Then, there only remains to decompose G_{exc} into at most βn cycles and β^{-2} exceptional edges for Lemma 5.3, or, $3\beta n$ paths for Lemma 5.4.

Recall that G_{exc} was introduced at the beginning of the proof. By construction, all edges of G_{exc} are exceptional and for any $x \in V_0$, if $xy, xy' \in E(G_{\text{exc}})$ are distinct then there exist distinct components C and C' of R such that $y \in V_G(C)$ and $y' \in V_G(C')$. Decompose G_{exc} into $s \leq \zeta n$ paths of length 2 with endpoints in $V \setminus V_0$ and an internal vertex in V_0 . Note that, by construction, each path has endpoints in clusters which lie in different connected components of R . Apply Lemma 4.16, with ε_9 and s playing the roles of ε and ℓ , and, each \mathcal{P}_i consisting of exactly one of the paths constructed above. We thus obtain $E^\diamond \subseteq E(\Gamma)$ such that $G_{\text{exc}} \cup E^\diamond$ admits a decomposition $\mathcal{D}'' \cup \mathcal{D}_{\text{exc}}$ where \mathcal{D}'' is a set of at most βn cycles and \mathcal{D}_{exc} is a set of at most β^{-2} exceptional edges. Add all edges in E^\diamond and G_{exc} to G' . By part (b) of Lemma 4.16, V_0, V_1, \dots, V_k is a $(\zeta, \beta, k, m, m', R')$ -superregular equalised partition of Γ .

Set $\tilde{\Gamma} := \Gamma$. For the proof of Lemma 5.3, add all cycles in \mathcal{D}'' to \mathcal{D} . By construction, $\mathcal{D} \cup \mathcal{D}_{\text{exc}}$ satisfies the desired properties. For the proof of Lemma 5.4, split each cycle in \mathcal{D}'' into two paths and add them to \mathcal{D} . Add the edges in \mathcal{D}_{exc} to \mathcal{D} . This completes the proof of Lemmas 5.3 and 5.4. \square

5.4. Covering the leftovers. This section corresponds to Step 4 of the proof overview. We will need the following fact.

Fact 5.7. *Assume G is Eulerian and $e \in E(G)$. Then G contains a cycle C such that $e \in C$.*

Lemma 5.8. *Suppose $0 < \frac{1}{n} \ll \frac{1}{k} \ll d \ll \beta \leq 1$ and $d' := d^{\frac{1}{10^5}}$. Let G and Γ be edge-disjoint graphs on vertex set V of size n such that $G \cup \Gamma$ is Eulerian and $\Delta(G) \leq dn$. Assume V_0, V_1, \dots, V_k is an (d, β, k, m, m', R) -superregular equalised partition of Γ such that any $x \in V \setminus V_0$ belongs to at least βk superregular pairs of Γ . Moreover, suppose that V_0 is a set of isolated vertices in G . Then there exists $E \subseteq E(\Gamma)$ such that $G \cup E$ can be decomposed into at most $2\beta n$ cycles. Moreover, V_0, V_1, \dots, V_k is a (d', β, k, m, m', R) -superregular equalised partition of $\Gamma \setminus E$.*

Proof. Fix an additional constant ζ such that $d \ll \zeta \ll \beta$. The idea is to decompose G into matchings and then apply Lemmas 4.12, 4.13, and 4.16 to tie the edges in each matching together to form cycles using Γ .

By Vizing's theorem (Theorem 4.17), we can decompose G into $\ell \leq dn+1 \leq 2dn$ matchings M_1, \dots, M_ℓ . Randomly split each matching M_i into $2\zeta^{-1}$ submatchings $M_{i,1}, \dots, M_{i,2\zeta^{-1}}$, by including each edge of M_i to $M_{i,j}$ independently with probability $\frac{\zeta}{2}$ for each $j \in [2\zeta^{-1}]$. By Lemma 3.1, we may assume that for each $i \in [\ell]$ and $i' \in [2\zeta^{-1}]$, we have $|M_{i,i'}| \leq \zeta n$ and for each $j \in [k]$, we have $|V(M_{i,i'}) \cap V_j| \leq \zeta m$. For simplicity, set $\ell' := \frac{2\ell}{\zeta} \leq \zeta n$ and relabel $M_{1,1}, \dots, M_{1,2\zeta^{-1}}, \dots, M_{\ell,1}, \dots, M_{\ell,2\zeta^{-1}}$ to $M_1, \dots, M_{\ell'}$. We successively apply Lemmas 4.12, 4.13, and 4.16, starting with $d, M_1, \dots, M_{\ell'}$ playing the roles of $\varepsilon, \mathcal{P}_1, \dots, \mathcal{P}_\ell$ in Lemma 4.12. We thus obtain $E_1 \subseteq E(\Gamma)$ such that $G \cup E_1$ admits a decomposition $\mathcal{D} \cup \mathcal{D}'$ where \mathcal{D} is a set of at most βn cycles and \mathcal{D}' is a set of at most β^{-2} edges. Moreover, by Lemma 4.2 and part (b) of Lemmas 4.12, 4.13, and 4.16, V_0, V_1, \dots, V_k is a $(d^{\frac{1}{2 \cdot 10^4}}, \beta, k, m, m', R)$ -superregular equalised partition of $\Gamma \setminus E_1$. By Fact 5.7 and Lemma 4.2, there exists $E_2 \subseteq \Gamma \setminus E_1$ such that $E(\mathcal{D}') \cup E_2$ can be decomposed into at most β^{-2} cycles and V_0, V_1, \dots, V_k is a (d', β, k, m, m', R) -superregular equalised partition of $\Gamma \setminus (E_1 \cup E_2)$. Let $E := E_1 \cup E_2$. This completes the proof. \square

5.5. Fully decomposing Γ . This section corresponds to Step 5 of the proof overview.

Lemma 5.9. *Let $0 < \frac{1}{m} \ll \frac{1}{k} \ll \frac{1}{K} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{f} \ll d \ll \frac{1}{\ell}, \frac{1}{g} \ll 1$ and suppose that $\frac{K}{7}, \frac{2K}{f}, \frac{2K}{g}, \frac{q}{f}, \frac{m}{4\ell K}, \frac{fm}{qK}, \frac{4fK}{3g(g-1)}, \frac{\ell}{2} \in \mathbb{N}^*$. Let G be an Eulerian graph on n vertices. Assume that V_0, V_1, \dots, V_k is a partition of $V(G)$ into an exceptional set V_0 consisting of at most εn isolated vertices and k clusters V_1, \dots, V_k of size m such that the corresponding reduced graph R of G is a cycle of even length, and for each $ij \in E(R)$, the pair $G[V_i, V_j]$ is $[\varepsilon, d]$ -superregular. Then G admits a cycle decomposition \mathcal{D} of size at most $dm + \varepsilon^{\frac{1}{16}}m$.*

To prove Lemma 5.9, we will use the robust decomposition lemma (Lemma 4.22). In order to apply this result, m needs to satisfy certain divisibility conditions and we need to find several refinements of the partition V_0, V_1, \dots, V_k . This would not be possible if, for example, m was prime. This explains why it is necessary to introduce the parameters K, q, f, ℓ , and g in the statement of Lemma 5.9.

Corollary 5.10. *Let $0 < \frac{1}{m} \ll \frac{1}{k} \leq \frac{1}{L} \ll \frac{1}{K} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{f} \ll d \ll \frac{1}{\ell}, \frac{1}{g} \ll 1$ and suppose that $\frac{K}{7}, \frac{2K}{f}, \frac{2K}{g}, \frac{q}{f}, \frac{m'}{4\ell K}, \frac{fm'}{qK}, \frac{4fK}{3g(g-1)}, \frac{\ell}{2} \in \mathbb{N}^*$. Let G be an n -vertex Eulerian graph and assume V_0, V_1, \dots, V_k is an $(\varepsilon, d, k, m, m', R)$ -superregular equalised partition of G such that V_0 is a set of isolated vertices in G . Assume R admits a decomposition \mathcal{D}_R satisfying the following properties. \mathcal{D}_R consists of at most $\frac{k}{2}$ cycles whose lengths are even and at least L . Moreover, for any distinct $i, j, j' \in [k]$, if $V_j V_i V_{j'}$ is a subpath of a cycle in \mathcal{D}_R , then the support clusters of V_i with respect to V_j and $V_{j'}$ are the same. Then G admits a cycle decomposition \mathcal{D} of size at most $\frac{dn}{2} + \varepsilon^{\frac{1}{33}}n$.*

Proof. Let $\mathcal{D} := \emptyset$. First apply Lemma 4.20 and add the cycles obtained to \mathcal{D} and delete their edges from G . Then, for each cycle $C = V_{i_1} \dots V_{i_{k'}}$ in \mathcal{D}_R , apply Lemma 5.9 with $2\sqrt{\varepsilon}, V_{i_1}, \dots, V_{i_{k'}}, i_{k'}, m', |V_0| + i_{k'}m'$ and $G[V_0 \cup V_G(C)]$ playing the roles of $\varepsilon, V_1, \dots, V_k, k, m, n$ and G , respectively, and add the cycles obtained to \mathcal{D} . \square

To prove Lemma 5.9, we will find an approximate decomposition of G using Lemma 5.12 and cover the leftover using the robust decomposition lemma (introduced in Section 4.4). The approximate decomposition will be obtained by repeatedly applying the following lemma, which is a special case of [27, Lemma 6.4].

Lemma 5.11. *Let $0 < \frac{1}{m} \ll d' \ll \varepsilon \ll d \ll \zeta, \frac{1}{t} \leq \frac{1}{2}$ and $k \geq 3$. Let G be a graph and V_1, \dots, V_k be a partition of $V(G)$ into k clusters of size m . Suppose that the following hold.*

- For each $i \in [k-1]$, $G[V_i, V_{i+1}]$ is a perfect matching M_i .
- $G[V_1, V_k]$ is $(\varepsilon, d', \zeta d', \frac{td'}{d})$ -superregular.

Then, $G[V_1, V_k]$ contains a perfect matching M such that $M \cup \left(\bigcup_{i=1}^{k-1} M_i\right)$ is a Hamilton cycle of G .

Lemma 5.12. *Suppose $0 < \frac{1}{m} \ll \frac{1}{k} \ll d' \ll \varepsilon \ll d \leq 1$. Let*

$$r := \frac{18d'm}{d} \quad \text{and} \quad h := dm - r,$$

and assume that $r, \frac{h}{k}, dm \in \mathbb{N}^$. Let G be an n -vertex graph with vertex set V . Assume that V_0, V_1, \dots, V_k is a partition of V into an exceptional set V_0 consisting of at most εn isolated vertices and k clusters V_1, \dots, V_k of size m such that the corresponding reduced graph R of G is a cycle, and, for each $ij \in E(R)$, the pair $G[V_i, V_j]$ is ε -regular and dm -regular. Then there exists $H \subseteq G$ such that, for each $ij \in E(R)$, $H[V_i, V_j]$ is r -regular and $G' := G \setminus H$ admits a decomposition \mathcal{D} into h Hamilton cycles of $G' - V_0$.*

Proof. Let H be the empty graph on V . Let

$$\varepsilon_1 := \varepsilon^{\frac{1}{12}}, \quad \varepsilon_2 := \varepsilon^{\frac{1}{25}}.$$

We may assume without loss of generality that $E(R) := \{V_i V_{i+1} \mid i \in [k]\}$, where $V_{k+1} := V_1$. For each $i \in [k]$, denote $G_i := G[V_i, V_{i+1}]$.

Let $i \in [k]$. Apply Lemma 4.7 to obtain an $(\varepsilon_1, d', \frac{d'}{2}, \frac{3d'}{2d})$ -superregular spanning subgraph $\Gamma_i \subseteq G_i$. Let $G'_i := G_i \setminus \Gamma_i$. One can easily verify that G'_i is ε_1 -regular and that, for each $x \in V_i \cup V_{i+1}$, we have $d_{G'_i}(x) = (d \pm \frac{3d'}{2d})m$.

In order to apply Lemma 5.11, we need to decompose each G'_i into perfect matchings. Thus, we will first ensure that the pairs G'_i are Eulerian and then apply Lemma 4.21 to regularise them.

Let $i \in [k]$. Apply Lemma 4.5 to obtain a Hamilton cycle $x_1 \dots x_{2m}$ of G'_i . Let $i_1 < \dots < i_\ell$ be the indices of the odd-degree vertices of G'_i . For each $s \in \{1, 3, \dots, \ell - 1\}$, add the edges of the path $x_{i_s} x_{i_s+1} \dots x_{i_{s+1}}$ to H and delete them from G'_i . By construction and Lemma 4.2, G'_i is now Eulerian and ε_2 -regular. Moreover, we have $d_{G'_i}(x) = (d \pm \frac{2d'}{d})m$ for each $x \in V_i \cup V_{i+1}$. Apply Lemma 4.21 to obtain a regular spanning subgraph G''_i of G'_i . By removing perfect matchings if necessary, we may assume G''_i is $(\frac{k-1}{k}h)$ -regular. Apply Hall's theorem to obtain a decomposition of G''_i into edge-disjoint sets \mathcal{D}_i^s , with $s \in [k] \setminus \{i\}$, each containing $\frac{h}{k}$ edge-disjoint perfect matchings. Add all edges of $G'_i \setminus G''_i$ to H .

Let $\ell \in [k]$. Let G^ℓ be the graph on vertex set $V \setminus V_0$ with $E(G^\ell) := (\bigcup_{i \in [k] \setminus \{\ell\}} E(\mathcal{D}_i^\ell)) \cup \Gamma_\ell$. We construct $\frac{h}{k}$ edge-disjoint Hamilton cycles $C_1, \dots, C_{\frac{h}{k}}$ of G^ℓ such that, for each $s \in [\frac{h}{k}]$ and $i \in [k] \setminus \{\ell\}$, C_s contains a perfect matching in \mathcal{D}_i^ℓ . In particular, observe that this implies that $C_s[V_\ell, V_{\ell+1}]$ is a perfect matching of $G^\ell[V_\ell, V_{\ell+1}] = \Gamma_\ell$.

Assume inductively that we have already constructed C_1, \dots, C_s for some $0 \leq s < \frac{h}{k}$. Delete from G^ℓ all edges of C_1, \dots, C_s . Note that since $\frac{h}{km} \ll \varepsilon_1, d'$, by Lemma 4.3, the pair $G^\ell[V_\ell, V_{\ell+1}]$ is still $(2\varepsilon_1, d', \frac{d'}{4}, \frac{3d'}{2d})$ -superregular. Let F be a spanning subgraph of G^ℓ such that, for each $i \in [k] \setminus \{\ell\}$, $F[V_i, V_{i+1}]$ is a perfect matching in \mathcal{D}_i^ℓ which has not been used for C_1, \dots, C_s and $F[V_\ell, V_{\ell+1}] = G^\ell[V_\ell, V_{\ell+1}]$. Then, Lemma 5.11 gives a Hamilton cycle C_{s+1} of $F \subseteq G^\ell$ satisfying the desired properties.

Proceed as above for each $\ell \in [k]$ and add all cycles obtained to \mathcal{D} . Add to H all remaining edges of $\bigcup_{i \in [k]} \Gamma_i$. This completes the proof. \square

We are now ready to prove Lemma 5.9, using the robust decomposition lemma. Recall the terminology introduced in Section 4.4.

Proof of Lemma 5.9. Let $\mathcal{D} := \emptyset$. We will repeatedly add cycles to \mathcal{D} . Whenever a cycle is added to \mathcal{D} , it is removed from G so that all cycles in \mathcal{D} are always pairwise edge-disjoint, as desired. In Steps 7 and 8, we will construct at most dm edge-disjoint Hamilton cycles of G . The additional cycles will be created during the regularising step (see Step 6).

Note that k is even. We may assume without loss of generality that $E(R) = \{V_i V_{i+1} \mid i \in [k]\}$, where $V_{k+1} := V_1$. For each $i \in [k]$, denote $G_i := G[V_i, V_{i+1}]$. Decompose R into two perfect matchings $M := \{V_i V_{i+1} \mid i \in [k]_{\text{odd}}\}$ and $M' := \{V_i V_{i+1} \mid i \in [k]_{\text{even}}\}$. As explained in Step 5 of the proof overview, the idea is to decompose each superregular pair of G into Hamilton paths

using the robust decomposition lemma and suitable fictive edges. We will then form Hamilton cycles of $G - V_0$ by tying together a Hamilton path of each pair in M using an edge from each pair in M' , and similarly for M and M' exchanged.

Step 1: Choosing the constants. Fix additional constants such that

$$0 < \frac{1}{m} \ll \frac{1}{k} \ll d'' \ll \frac{1}{K} \ll \varepsilon \ll \frac{1}{q} \ll \frac{1}{f} \ll \frac{r_1 K}{m} \ll d \ll \frac{1}{\ell}, \frac{1}{g} \ll 1.$$

Let

$$\varepsilon_1 := \varepsilon^{\frac{1}{12}}, \quad \varepsilon_2 := \varepsilon^{\frac{1}{84}}, \quad \varepsilon_3 := \varepsilon^{\frac{1}{588}}, \quad \varepsilon_4 := \varepsilon^{\frac{1}{1177}},$$

and,

$$\varepsilon_1^* := \varepsilon^{\frac{1}{3}}, \quad \varepsilon_2^* := \varepsilon^{\frac{1}{7}}, \quad \varepsilon_3^* := \varepsilon^{\frac{1}{15}}, \quad \varepsilon_4^* := \varepsilon^{\frac{1}{31}}, \quad \varepsilon_5^* := \varepsilon^{\frac{1}{125}}, \quad \varepsilon_6^* := \varepsilon^{\frac{1}{875}}, \quad \varepsilon_7^* := \varepsilon^{\frac{1}{1751}}.$$

Let $c := c(d, k)$ be the constant in Lemma 4.20 and define

$$d' := d - 11\varepsilon_2^*, \quad r := \frac{9d''m}{d'}.$$

Observe that $r(2K)^2 \leq \frac{m}{K}$. Let

$$r_2 := 192\ell g^2 K r, \quad r_3 := \frac{2rfK}{q}, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3, \quad s := 2rfK + 7r^\diamond.$$

Note that $r, r_2, r_3 \leq r_1$ and $r^\diamond \leq 2r_1$. By adjusting ε, d , and d'' slightly, we may assume that $\frac{(d-10\varepsilon_2^*)m}{2}, \frac{d'm}{2}, \frac{d'm-2r}{k}, r, r_3 \in \mathbb{N}^*$.

Step 2: Constructing the bi-setups. Let $i \in [k]$. Apply Corollary 4.9 to obtain an orientation \vec{G}_i of G_i such that both $\vec{G}_i[V_i, V_{i+1}]$ and $\vec{G}_i[V_{i+1}, V_i]$ are $[\varepsilon_1, \frac{d}{2}]$ -superregular (here and below, the index is taken modulo k).

For each $i \in [k]$, randomly partition V_i into K subclusters $V_{i,1}, \dots, V_{i,K}$ of equal size. This induces, for each $i \in [k]$, a partition \mathcal{P}_i of $V(\vec{G}_i)$ into $2K$ clusters of size $\frac{m}{K}$. By Lemma 4.10, we may assume that for each $i \in [k]$, the partition \mathcal{P}_i is an ε_2 -superregular K -refinement of $\{V_i, V_{i+1}\}$. Let \vec{R}_i be the reduced digraph of \vec{G}_i with respect to \mathcal{P}_i . (Thus, \vec{R}_i is the complete bipartite digraph with vertex classes of size K .) Proceed similarly to obtain, for each $i \in [k]$, an ε_3 -superregular ℓ -refinement \mathcal{P}'_i of \mathcal{P}_i . Let \vec{R}'_i be the reduced digraph of \vec{G}_i with respect to \mathcal{P}'_i .

For each $i \in [k]$, let $C_i := V_{i,1}V_{i+1,1}V_{i,2} \dots V_{i+1,K}$ and observe that C_i is a Hamilton cycle of \vec{R}_i . Thus, by construction, for each $i \in [k]$, $(\vec{G}_i, \mathcal{P}_i, \mathcal{P}'_i, \vec{R}_i, \vec{R}'_i, C_i)$ is an $(\ell, 2K, \frac{m}{K}, \varepsilon_3, \frac{d}{2})$ -bi-setup.

Step 3: Selecting the fictive edges. In this step, we will set aside a set \mathcal{E} of edges which will enable us to tie together the Hamilton path obtained with the robust decomposition lemma. Then, we will construct a corresponding set \mathcal{F} of fictive edges which will prescribe the endpoints of the Hamilton paths (recall Figure 1). These fictive edges will then be incorporated in the special path systems. Thus, in order to satisfy (SPS2), we will need to ensure that the endpoints of the edges in \mathcal{E} lie in the appropriate subclusters (see Figure 2). We start by choosing the fictive edges which will be included in the special factors required for finding the chord absorbers (see part (i) of Lemma 4.22).

First, proceed as in Step 2 to construct, for each $i \in [k]$, an ε_3 -superregular $\frac{q}{f}$ -refinement \mathcal{P}_i^* of \mathcal{P}_i . For each $i \in [k]$ and $V_{i,j} \in \mathcal{P}_i$, denote by $V_{i,j,1}, \dots, V_{i,j,\frac{q}{f}}$ the partition of $V_{i,j}$ induced by \mathcal{P}_i^* .

For each $i \in [k]$, denote by $\mathcal{I}_i := \{I_{i,1}, \dots, I_{i,f}\}$ the canonical interval partition of C_i into f intervals of length $\frac{2K}{f}$. For each $i \in [k]$, $j \in [f]$ and $h \in [\frac{q}{f}]$, apply Corollary 4.6 to obtain a set $\mathcal{E}_{i,j,h}^{CA}$ of r_3 vertex-disjoint edges of $\vec{G}_i[V_{i,j',h}, V_{i+1,j',h}]$, where $j' := \frac{jK}{f}$. Let $e_{i,j,h,1}, \dots, e_{i,j,h,r_3}$ be an enumeration of the edges in $\mathcal{E}_{i,j,h}^{CA}$.

We construct fictive edges as follows. Let $i \in [k]$, $j \in [f]$, and $h \in [\frac{q}{f}]$. For each $t \in [r_3]$, let $f_{i,j,h,t}$ be a fictive edge from x to y , where x is the endpoint of $e_{i-1,j,h,t}$ which belongs to V_i and y is the endpoint of $e_{i+1,j,h,t}$ which belongs to V_{i+1} . Let $\mathcal{F}_{i,j,h}^{CA}$ be the set of all these fictive edges. Observe that each edge in $\mathcal{F}_{i,j,h}^{CA}$ will be a suitable fictive edge for a special path system of style h spanning the interval $I_{i,j}$. Indeed, by construction of $\mathcal{E}_{i-1,j,h}^{CA}$ and $\mathcal{E}_{i+1,j,h}^{CA}$, $\mathcal{F}_{i,j,h}^{CA} \subseteq E(\vec{G}_i[V_{i,j',h}, V_{i+1,j',h}])$, where $j' := \frac{jK}{f}$. I.e. each edge in $\mathcal{F}_{i,j,h}^{CA}$ lies in the “ h^{th} subpair” of the penultimate pair along the interval $I_{i,j}$, as desired for (SPS2). Let \mathcal{E}^{CA} be the union of the sets $\mathcal{E}_{i,j,h}^{CA}$ for each $i \in [k]$, $j \in [f]$, and $h \in [\frac{q}{f}]$. Let \mathcal{E}_i^{CA} be the union of the sets $\mathcal{E}_{i,j,h}^{CA}$ for each $j \in [f]$ and $h \in [\frac{q}{f}]$. Define \mathcal{F}^{CA} and \mathcal{F}_i^{CA} similarly.

Note that for all $j \in [f]$, $h \in [\frac{q}{f}]$, and $t \in [r_3]$, the graph $(\bigcup_{i \in [k]_{\text{odd}}} f_{i,j,h,t}) \cup (\bigcup_{i \in [k]_{\text{even}}} e_{i,j,h,t})$ is a (directed) cycle of length k which intersects each of the clusters V_1, \dots, V_k . The same holds with $[k]_{\text{odd}}$ and $[k]_{\text{even}}$ exchanged. Therefore, in particular, the following property is satisfied.

- $\mathcal{E}^{CA} \cup \mathcal{F}^{CA}$ can be decomposed into edge-disjoint (directed) cycles of length k , each containing either
- an edge of \mathcal{E}^{CA} between V_i and V_{i+1} for each $i \in [k]_{\text{odd}}$ and an edge of \mathcal{F}^{CA} between V_i and V_{i+1} for each $i \in [k]_{\text{even}}$, or
 - an edge of \mathcal{F}^{CA} between V_i and V_{i+1} for each $i \in [k]_{\text{odd}}$ and an edge of \mathcal{E}^{CA} between V_i and V_{i+1} for each $i \in [k]_{\text{even}}$.

Property (††) will eventually enable us to construct Hamilton cycles of $G - V_0$ by tying together a Hamilton path of each pair in M using an edge from each pair in M' , or vice versa (recall Figure 1).

We now select the fictive edges which will be included in the special factors required for finding the parity chord absorbers (see part (ii) of Lemma 4.22). We proceed as above to construct, for each $i \in [k]$ and $j \in [7]$, a set $\mathcal{E}_{i,j}^{PCA}$ of $5r^\diamond$ vertex-disjoint edges of $\vec{G}_i[V_{i,j'}, V_{i+1,j'}]$, where $j' := \frac{jK}{7}$. For each $i \in [k]$, since \mathcal{E}_i^{CA} is a matching, by Lemma 4.2, we can ensure that \mathcal{E}_i^{CA} and $\mathcal{E}_{i,j}^{PCA}$ are edge-disjoint. Let \mathcal{E}^{PCA} be the union of the sets $\mathcal{E}_{i,j}^{PCA}$, for $i \in [k]$ and $j \in [7]$. For each $i \in [k]$ and $j \in [7]$, we construct a set $\mathcal{F}_{i,j}^{PCA}$ of fictive edges as above and let \mathcal{F}^{PCA} be the union of these sets. Importantly, the edges in \mathcal{E}^{PCA} and \mathcal{F}^{PCA} satisfy (the analogue of) property (††).

Define $\mathcal{E} := \mathcal{E}^{CA} \cup \mathcal{E}^{PCA}$ and $\mathcal{F} := \mathcal{F}^{CA} \cup \mathcal{F}^{PCA}$. Delete from \vec{G} (and G) all the edges in \mathcal{E} . For each $i \in [k]$, note that we have deleted from \vec{G}_i at most 2 edges incident to each vertex so, by Lemma 4.2, $(\vec{G}_i, \mathcal{P}_i, \mathcal{P}'_i, \vec{R}_i, \vec{R}'_i, C_i)$ is still an $(\ell, 2K, \frac{m}{K}, \varepsilon_4, \frac{d}{2})$ -bi-setup and \mathcal{P}_i^* is still a ε_4 -superregular $\frac{q}{f}$ -refinement of \mathcal{P}_i .

Step 4: Constructing the special factors. In this step, we will construct, for $i \in [k]$, edge-disjoint special factors $SF_{i,1}, \dots, SF_{i,r_3}$ with parameters $(\frac{q}{f}, f)$ with respect to C_i, \mathcal{P}_i^* in \vec{G}_i such that, for all $t \in [r_3]$, $\text{Fict}(SF_{i,t}) = \{f_{i,j,h,t} \mid j \in [f], h \in [\frac{q}{f}]\}$. (Recall Figure 2.)

Let $i \in [k]$, $j \in [f]$, and $h \in [\frac{q}{f}]$. Suppose inductively that, for some $0 \leq t \leq r_3$, we have constructed edge-disjoint special path systems $SPS_{i,j,h,1}, \dots, SPS_{i,j,h,t}$ of style h in \vec{G}_i spanning the interval I_j and such that, for each $i' \in [t]$, $f_{i,j,h,i'}$ is the fictive edge contained in $SPS_{i,j,h,i'}$. If $t < r_3$, we construct $SPS_{i,j,h,t+1}$ as follows. For simplicity, denote $I_{i,j} = U_1 \dots U_{\frac{2K}{f}+1}$ and, for each $i' \in [\frac{2K}{f}+1]$, let $U_{i',h}$ denote the h^{th} subcluster of $U_{i'}$ in \mathcal{P}_i^* . Let $\vec{G}'_i := \vec{G}_i \setminus \bigcup_{i' \in [t]} SPS_{i,j,h,i'}$. By Lemma 4.2 and since $r_3 \leq \varepsilon_4 \frac{fm}{qK} = \varepsilon_4 |U_{i',h}| = \varepsilon_4 |U_{i'+1,h}|$, $\vec{G}'_i[U_{i',h}, U_{i'+1,h}]$ is $[2\sqrt{\varepsilon_4}, \geq \frac{d}{2}]$ -superregular for each $i' \in [\frac{2K}{f}]$ and $\vec{G}'_i[U_{\frac{2K}{f}-1,h} \setminus V(f_{i,j,h,t+1}), U_{\frac{2K}{f},h} \setminus V(f_{i,j,h,t+1})]$ is $[2\sqrt{\varepsilon_4}, \geq \frac{d}{2}]$ -superregular. For each $i' \in [\frac{2K}{f}] \setminus \{\frac{2K}{f} - 1\}$, apply Corollary 4.6 with $\vec{G}'_i[U_{i',h}, U_{i'+1,h}]$, $2\sqrt{\varepsilon_4}$, and $\frac{d}{3}$ playing the roles of G, ε , and α to obtain a perfect matching $M_{i'}$ in $\vec{G}'_i[U_{i',h}, U_{i'+1,h}]$. Apply

Corollary 4.6 with $\vec{G}'_i[U_{\frac{2K}{f}-1,h} \setminus V(f_{i,j,h,t+1}), U_{\frac{2K}{f},h} \setminus V(f_{i,j,h,t+1})]$, $2\sqrt{\varepsilon_4}$, and $\frac{d}{3}$ playing the roles of G , ε , and α to obtain a perfect matching $M'_{\frac{2K}{f}-1}$ in $\vec{G}'_i[U_{\frac{2K}{f}-1,h} \setminus V(f_{i,j,h,t+1}), U_{\frac{2K}{f},h} \setminus V(f_{i,j,h,t+1})]$. Let $M_{\frac{2K}{f}-1} := M'_{\frac{2K}{f}-1} \cup \{f_{i,j,h,t+1}\}$. Then, $SPS_{i,t+1} := \bigcup_{i' \in [\frac{2K}{f}]} M_{i'}$ is a special path system of style h in \vec{G}_i spanning the interval I_j which is edge-disjoint from $SPS_{i,1}, \dots, SPS_{i,t}$ and which contains the fictive edge $f_{i,j,h,t+1}$.

Thus, we can construct, for each $i \in [k]$, $j \in [f]$, and $h \in [\frac{q}{f}]$, edge-disjoint special path systems $SPS_{i,j,h,1}, \dots, SPS_{i,j,h,r_3}$ of style h in \vec{G}_i spanning the interval I_j such that, for each $t \in [r_3]$, $f_{i,j,h,t}$ is the fictive edge contained in $SPS_{i,j,h,t}$. For each $i \in [k]$ and $t \in [r_3]$, let $SF_{i,t} := \bigcup_{j \in [f]} \bigcup_{h \in [\frac{q}{f}]} SPS_{i,j,h,t}$. Then, for each $i \in [k]$, $SF_{i,1}, \dots, SF_{i,r_3}$ are edge-disjoint special factors with parameters $(\frac{q}{f}, f)$ with respect to C_i , \mathcal{P}_i^* in \vec{G}_i such that, for all $t \in [r_3]$, $\text{Fict}(SF_{i,t}) = \{f_{i,j,h,t} \mid j \in [f], h \in [\frac{q}{f}]\}$. For each $i \in [k]$, let $\mathcal{SF}_i := SF_{i,1} \cup \dots \cup SF_{i,r_3}$.

Step 5: Finding the robustly decomposable digraphs. For any $i \in [k]$, we apply Lemma 4.22 with $\frac{m}{K}$, $2K$, ε_4 , $\frac{d}{2}$, \vec{G}_i , \mathcal{P}_i , \mathcal{P}'_i , \vec{R}_i , \vec{R}'_i , C_i , and \mathcal{P}_i^* playing the roles of m , k , ε , d , \vec{G} , \mathcal{P} , \mathcal{P}' , \vec{R} , \vec{R}' , C , and \mathcal{P}^* to obtain a digraph $\vec{CA}_i(r)$ satisfying the properties described in Lemma 4.22.

Since $\vec{CA}_i(r) \cup \mathcal{SF}_i$ is $(r_1 + r_2 + r_3)$ -regular and $r_1 + r_2 + r_3 \leq 3r_1$, Lemma 4.2 implies that one can proceed similarly as in Step 4 to construct special factors $SF'_{i,1}, \dots, SF'_{i,r^\diamond}$ with parameters $(1, 7)$ with respect to C_i, \mathcal{P}_i in \vec{G}_i which are edge-disjoint from each other and from $\vec{CA}_i(r) \cup \mathcal{SF}_i$.

Let $\vec{PCA}_i(r)$ and \vec{G}_i^{rob} be as in Lemma 4.22. For each $i \in [k]$, delete the edges of \vec{G}_i^{rob} from G_i (and G). Since \vec{G}_i^{rob} is $(r_1 + r_2 + r_3 + 5r^\diamond + r^\diamond)$ -regular, we have deleted at most $2(r_1 + r_2 + r_3 + 5r^\diamond + r^\diamond) \leq 30r_1 \leq \varepsilon m$ edges incident to each vertex in $V_i \cup V_{i+1}$. Moreover, recall that, at the end of Step 3, we have already deleted from G the edges in \mathcal{E} , which contains at most two edges incident to each vertex in G_i , for each $i \in [k]$. Thus, Lemma 4.2 implies that G_i is still $[\varepsilon_1^*, d]$ -superregular. Furthermore, $(\dagger\dagger)$ and its analogue for \mathcal{E}^{PCA} and \mathcal{F}^{PCA} ensure that G is still Eulerian.

Step 6: Regularising the superregular pairs. In order to apply Lemma 5.12, we need to regularise each superregular pair of G . We will first apply the tools of Section 4.3 to each G_i separately, but, we will see that this yields too many cycles. We will therefore use a few further edges of G to tie together some of the cycles obtained to form longer cycles. We make sure that, when tying some of the cycles together, we use only a bounded number of edges incident to each vertex. Thus, applying the tools of Section 4.3 once again will only yield a few additional cycles.

First, apply Lemma 4.20 (to the current graph G) with ε_1^* and m playing the roles of ε and m' . Add the resulting cycles to \mathcal{D} and delete their edges from G . Note that we have added at most c cycles to \mathcal{D} . Moreover, for each $i \in [k]$, the pair G_i is now Eulerian and $[\varepsilon_2^*, d]$ -superregular.

For each $i \in [k]$, apply Lemma 4.21 with G_i, ε_2^* , and $2\varepsilon_2^*m$ playing the roles of G, ε , and Θ in order to obtain a set \mathcal{C}_i of at most $4\varepsilon_2^*m$ edge-disjoint cycles of length at least $\frac{2m}{3}$ such that the following holds. Delete the edges of \mathcal{C}_i from G_i . Then, G_i is regular and $[\varepsilon_3^*, d]$ -superregular. By adding additional edge-disjoint Hamilton cycles to \mathcal{C}_i if necessary, we may assume that G_i is $(d - 10\varepsilon_2^*)m$ -regular and $|\mathcal{C}_i| \leq 10\varepsilon_2^*m$. We observe that $\bigcup_{i \in [k]} \mathcal{C}_i$ may contain up to $10\varepsilon_2^*mk$ cycles, so we need to split each of these cycles into paths and tie them together to form fewer cycles.

Let $i \in [k]$. Split one by one each cycle in \mathcal{C}_i into at most $\frac{30}{d}$ paths of length at most $\frac{dm}{10}$, each with an endpoint in V_i and an endpoint in V_{i+1} , and such that each vertex in $V_i \cup V_{i+1}$ is an endpoint of at most 2 paths. This is possible since the cycles in \mathcal{C}_i have length at least $\frac{2m}{3}$ while, on the other hand, in each step there are at most $\frac{30}{d}|\mathcal{C}_i| \leq \varepsilon_3^*m$ vertices in each cluster which are already endpoints of 2 paths. Let \mathcal{P}_i be the set of paths obtained at the end of this procedure and observe that $|\mathcal{P}_i| \leq \varepsilon_3^*m$.

Decompose $\bigcup_{i \in [k]_{\text{odd}}} \mathcal{P}_i$ into at most $\varepsilon_3^* m$ sets of paths, each containing at most one path in \mathcal{P}_i for each $i \in [k]_{\text{odd}}$. Decompose $\bigcup_{i \in [k]_{\text{even}}} \mathcal{P}_i$ similarly. Let $\mathcal{P}'_1, \dots, \mathcal{P}'_{\ell'}$ be the sets of paths obtained. Thus, $\ell' \leq 2\varepsilon_3^* m$. Apply Lemma 4.19 with $\varepsilon_3^*, 2\varepsilon_3^*, \ell'$, and $\mathcal{P}'_1, \dots, \mathcal{P}'_{\ell'}$ playing the roles of ε, ζ, ℓ , and $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ to obtain $E \subseteq E(G)$ such that $(\mathcal{P}'_1 \cup \dots \cup \mathcal{P}'_{\ell'}) \cup E$ can be decomposed into ℓ' cycles. Add these cycles to \mathcal{D} and delete from G all the edges in E . Note that, for each $i \in [k]$, by Lemma 4.2 and part (a) of Lemma 4.19, G_i is now $[\varepsilon_4^*, d]$ -superregular with maximum degree at most $(d - 10\varepsilon_2^*)m$ and minimum degree at least $(d - 10\varepsilon_2^*)m - 6$.

We now need to regularise the superregular pairs of G once again. First, we apply Lemma 4.20 and add the resulting cycles to \mathcal{D} . Then, we apply Lemma 4.21 to G_i , for each $i \in [k]$, and add all cycles obtained to \mathcal{D} . Using similar arguments as above, we may assume that, for each $i \in [k]$, the pair G_i is now $[\varepsilon_5^*, d]$ -superregular and d' -regular. We note that $|\mathcal{D}| \leq 3\varepsilon_3^* m \leq \varepsilon_{16}^* m$, as desired.

Step 7: Approximately decomposing (the remainder of) G . Apply Lemma 5.12 with ε_5^*, d', d'' , and $2r$ playing the roles of ε, d, d' , and r to obtain $H \subseteq G$ such that, for each $i \in [k]$, $H_i := H[V_i, V_{i+1}]$ is $2r$ -regular and $G \setminus H$ can be decomposed into $d'm - 2r$ cycles. Add these cycles to \mathcal{D} .

Step 8: Decomposing the leftover and robustly decomposable graphs. Let $i \in [k]$. Since H_i is $2r$ -regular, there exists an orientation \vec{H}_i of H_i such that $\vec{H}_i[V_i \cup V_{i+1}]$ is an r -regular bipartite oriented graph with vertex classes V_i and V_{i+1} . Let \mathcal{D}_i be the Hamilton decomposition of $\vec{H}_i \cup \vec{G}_i^{\text{rob}}$ guaranteed by Lemma 4.22. Note that, in particular, each cycle in \mathcal{D}_i contains exactly one fictive edge and thus corresponds to a Hamilton path of the original graph $G[V_i, V_{i+1}]$. Moreover, $|\mathcal{D}_i| = s$.

We form $2s$ cycles by removing the fictive edges in \mathcal{F} and inserting back the edges in \mathcal{E} as follows. Fix a decomposition of $\mathcal{E} \cup \mathcal{F}$ into edge-disjoint cycles of length k satisfying the property described in (††). Let C be a cycle in this decomposition and assume without loss of generality that the fictive edges in C lie between V_i and V_{i+1} for $i \in [k]_{\text{odd}}$. Let $f_1, e_2, f_3, \dots, e_k$ be an enumeration of the edges of C where, for each $i \in [k]$, the edge f_i (respectively e_i) lies between V_i and V_{i+1} . For each $i \in [k]_{\text{odd}}$, let $C_i \in \mathcal{D}_i$ be the cycle which contains f_i . Then, by construction, $(\bigcup_{i \in [k]_{\text{odd}}} C_i \setminus \{f_i\}) \cup (\bigcup_{i \in [k]_{\text{even}}} e_i)$ is a cycle and we add this cycle to \mathcal{D} . We proceed in this way for every cycle C in the cycle decomposition of $\mathcal{E} \cup \mathcal{F}$. This gives a cycle decomposition \mathcal{D} of our original graph G of size at most $(d'm - 2r) + 2s + \varepsilon_{16}^* m \leq dm + \varepsilon_{16}^* m$. \square

5.6. Proof of the main theorems. We are now ready to prove Theorems 1.10(i), 1.10(ii), 1.11 and 1.13.

Proof of Theorem 1.10(ii). Let $\mathcal{D} := \emptyset$. We will repeatedly add cycles to \mathcal{D} . The set \mathcal{D} will eventually be the set of cycles for our final decomposition of G . The proof is structured as described in Section 2.

Fix additional constants such that $0 < \frac{1}{n_0} \ll \frac{1}{M} \ll \varepsilon \ll \zeta \ll d \ll \beta \ll \alpha, \delta \leq 1$ and $0 < \frac{1}{M} \leq \frac{1}{L} \ll \frac{1}{K} \ll d \ll \frac{1}{q} \ll \frac{1}{f} \ll \beta \ll \frac{1}{\ell}, \frac{1}{g} \ll 1$, with $\frac{K}{7}, \frac{2K}{f}, \frac{2K}{g}, \frac{q}{f}, \frac{4fK}{3g(g-1)}, \frac{\ell}{2} \in \mathbb{N}^*$. Let G be a graph on $n \geq n_0$ vertices with $\delta(G) \geq \alpha n$. Let $V := V(G)$ and

$$\varepsilon' := \varepsilon^{\frac{1}{75}}, \quad \zeta' := \zeta^{\frac{1}{3}}, \quad d_1 := d^{\frac{1}{10^7}}, \quad d_2 := d_1^{\frac{1}{33}}.$$

Step 1: Applying Szemerédi's regularity lemma and setting aside some random subgraphs Γ and Γ' . Apply Lemma 5.1 with parameters $M, L, \varepsilon, \zeta, d, \beta, \alpha$ and with $4q\ell K$ playing the role of r to obtain parameters $M', m' \in \mathbb{N}^*$, a decomposition of G into four edge-disjoint graphs G^*, Γ, Γ' , and H , and a partition of V into k clusters V_1, \dots, V_k and an exceptional set V_0 satisfying the properties described in Lemma 5.1. In particular, the following property is

satisfied.

(‡) *The reduced graph R' of Γ admits a decomposition $\mathcal{D}_{R'}$ such that the following hold. $\mathcal{D}_{R'}$ consists of at most $\frac{k}{2}$ cycles whose lengths are even and at least L . Moreover, for any distinct $i, j, j' \in [k]$, if $V_j V_i V_{j'}$ is a subpath of a cycle in $\mathcal{D}_{R'}$ then the support clusters of V_i with respect to V_j and $V_{j'}$ are the same.*

Step 2: Covering the edges of $G[V_0]$. Apply Lemma 5.2 with G^* playing the role of G to obtain a graph $H_0 \subseteq G^* \cup \Gamma$ satisfying properties (a)–(d) of Lemma 5.2. In particular, there exists a decomposition $\mathcal{D}_0 \cup \mathcal{D}'_0$ of H_0 such that \mathcal{D}_0 is a set of at most βn cycles and \mathcal{D}'_0 is a set of at most β^{-2} edges. Add the cycles in \mathcal{D}_0 to \mathcal{D} . Since G is Eulerian, by Fact 5.7, we can cover the edges in \mathcal{D}'_0 with at most β^{-2} edge-disjoint cycles. Add these cycles to \mathcal{D} and delete the edges in all these cycles from G, G^*, Γ , and Γ' . Observe that by Lemmas 4.2 and 5.2, V_0, V_1, \dots, V_k is

- an $(\varepsilon', \geq d, k, m, m', R)$ -superregular equalised partition of G^* ;
- an $(\varepsilon', \beta, k, m, m', R')$ -superregular equalised partition of Γ ; and
- an $(\varepsilon', \zeta, k, m, m', R'')$ -superregular equalised partition of Γ' ,

where R' and R'' are edge-disjoint and satisfy $R' \cup R'' = R$. Moreover, $G[V_0]$ is now empty, as desired.

Step 3: Covering most of G^* with at most roughly $\frac{n}{2}$ cycles. We now apply Lemma 5.3 with G^* and ε' playing the roles of G and ε to obtain a decomposition of $G^* \cup \Gamma \cup \Gamma'$ into edge-disjoint graphs $G', \tilde{\Gamma}$, and H' such that $G^*, \Gamma' \subseteq G' \cup H'$, $\tilde{\Gamma} \subseteq \Gamma$, and properties (a)–(c) of Lemma 5.3 are satisfied. In particular, there exists a decomposition $\mathcal{D}' \cup \mathcal{D}'_{\text{exc}}$ of G' such that \mathcal{D}' is a set of at most $\frac{n}{2} + 2\beta n$ cycles and $\mathcal{D}'_{\text{exc}}$ is a set of at most β^{-2} edges. Add all cycles in \mathcal{D}' to \mathcal{D} . Apply Fact 5.7 with $\tilde{\Gamma} \cup H \cup H' \cup \mathcal{D}'_{\text{exc}}$ playing the role of G to cover the edges in $\mathcal{D}'_{\text{exc}}$ with at most β^{-2} edge-disjoint cycles. Add these cycles to \mathcal{D} and delete the edges in all these cycles from $\tilde{\Gamma}, H$, and H' . By Lemmas 4.2 and 5.3, V_0, V_1, \dots, V_k is a $(\zeta', \beta, k, m, m', R)$ -superregular equalised partition of $\tilde{\Gamma}$. Also note that $\Delta(H \cup H') \leq 4dn + 13\zeta n \leq 5dn$.

Step 4: Covering the leftovers. We now cover the edges of $H \cup H'$ by applying Lemma 5.8 with $H \cup H', \tilde{\Gamma}$, and $5d$ playing the roles of G, Γ , and d to obtain a subgraph $\tilde{H} \subseteq \tilde{\Gamma}$ and a decomposition $\tilde{\mathcal{D}}$ of $H \cup H' \cup \tilde{H}$ into at most $2\beta n$ cycles. Add all cycles in $\tilde{\mathcal{D}}$ to \mathcal{D} and let $\tilde{\Gamma}' := \tilde{\Gamma} \setminus \tilde{H}$. By Lemma 5.8, V_0, V_1, \dots, V_k is a $(d_1, \beta, k, m, m', R')$ -superregular equalised partition of $\tilde{\Gamma}'$.

Step 5: Fully decomposing Γ . Finally, observe that $\tilde{\Gamma}'$ is an Eulerian subgraph of Γ with the same reduced graph and the same support clusters, so property (‡) holds for $\tilde{\Gamma}'$. Thus, we can apply Corollary 5.10 with $\tilde{\Gamma}', R', d_1$, and β playing the roles of G, R, ε , and d to obtain a decomposition of $\tilde{\Gamma}'$ into at most $\frac{\beta n}{2} + d_2 n$ cycles. Add these cycles to \mathcal{D} . Then, \mathcal{D} forms a cycle decomposition of G and $|\mathcal{D}| \leq \frac{n}{2} + \delta n$, as desired. \square

Proof of Theorem 1.10(i). We modify the proof of Theorem 1.10(ii) to get a path decomposition as follows. Step 1 is identical. For Step 2, we simply apply Theorem 1.1 to obtain a path decomposition of $G[V_0]$ into at most εn paths. For Step 3, first remove an edge incident to each exceptional vertex of odd degree in G^* so that property (vi) of Lemma 5.4 holds. We view these edges as individual paths in our decomposition. We can thus apply Lemma 5.4 instead of Lemma 5.3, with the set of odd degree vertices of $G^* \cup H \cup \Gamma \cup \Gamma'$ playing the role of U . Then, by Lemma 5.4(c'), $H \cup H' \cup \tilde{\Gamma}$ is Eulerian at the end of Step 3. Thus, we can apply the arguments of Steps 4 and 5 and split each cycle obtained in these steps into two paths in order to obtain a path decomposition of $H \cup H' \cup \tilde{\Gamma}$. One can easily verify that we obtain at most $\frac{n}{2} + \delta n$ paths in total. \square

The weak quasirandomness assumed in the next two proofs allows for a more efficient decomposition. The critical property implied by weak quasirandomness is that the reduced graph R is connected.

Proof of Theorem 1.11. We use the same arguments as in the proof of Theorems 1.10(i) and 1.10(ii) with $\beta \ll \alpha, \delta, p$ and applying Lemma 5.5 instead of Lemmas 5.3 and 5.4. This is possible since the reduced graph R of G^* is connected.

Indeed, assume for a contradiction that R is disconnected and let C be a component of R . Let $A := V_{G^*}(C)$ and $B := V(G) \setminus A$. Since $\delta(G) \geq \alpha n$, it is easy to see that $|A|, |B| \geq \frac{\alpha n}{2}$. But, by Lemma 5.1,

$$\begin{aligned} e_G(A, B) &= e_H(A, B) + e_\Gamma(A, B) + e_{\Gamma'}(A, B) + e_{G^*}(A, V_0) \\ &\leq |A|(4dn + (\beta + \varepsilon)n + (\zeta + \varepsilon)n + \varepsilon n) \\ &< p|A||B|, \end{aligned}$$

contradicting the fact that G is weakly- $(\frac{\alpha}{2}, p)$ -quasirandom.

Also observe that in the path decomposition case, we have $\text{odd}(G^* \cup H \cup \Gamma \cup \Gamma') \leq \text{odd}(G) + |V_0|$ at the point where we apply Lemma 5.5 (recall the proof of Theorem 1.10(i)), so we obtain a decomposition of the desired size. \square

Proof of Theorem 1.13. First observe that since G is weakly- (ε, p) -quasirandom, G has fewer than εn vertices of degree less than $\frac{pn}{2}$. Let X be the set of these vertices. We modify the proof Theorem 1.11(i) as follows. Apply the arguments of Step 1 with $G - X$ and $\frac{p}{3}$ playing the roles of G and α . Add the vertices in X to the exceptional set and all edges incident to these vertices to G^* . The remainder of the proof is identical. \square

6. CONCLUDING REMARKS

We conclude by deriving Theorem 1.10(iii) and providing some remarks on our results.

6.1. Proof of Theorem 1.10(iii). We now show how Theorem 1.10(iii) can be derived from Theorem 1.10(ii). Let G be a graph. We saw in the introduction that one can remove at most $n - 1$ edges of G to obtain an Eulerian graph. However, in order to apply Theorem 1.10(ii), we also need to make sure that the resulting Eulerian graph still has linear minimum degree.

Proof of Theorem 1.10(iii). Fix $\ell := \text{odd}(G)$. Let V_{odd} be the set of odd-degree vertices of G . We repeatedly remove short paths with endpoints in V_{odd} (but ℓ is left unchanged). Fix a maximum matching M of $G[V_{\text{odd}}]$. Delete the edges of M from G and remove the vertices in $V(M)$ from V_{odd} . We observe that V_{odd} is now an independent set of G .

If there exists a path xyz in G such that $x, z \in V_{\text{odd}}$ are distinct and fewer than $\frac{\alpha n}{4}$ edges incident to y have been deleted so far, remove the edges xy and yz from G and the vertices x, z from V_{odd} . We repeat this procedure until there exists no such path of length 2.

Then, we claim that $|V_{\text{odd}}| \leq \frac{2}{\alpha}$. Indeed, at each stage, there are at most $\frac{8}{\alpha}$ vertices $y \in V \setminus V_{\text{odd}}$ such that we have deleted at least $\frac{\alpha n}{4}$ edges incident to y . By construction, for each $x \in V_{\text{odd}}$, no edge incident to x has been deleted from G and, thus, x has more than $\frac{\alpha n}{2}$ neighbours y such that fewer than $\frac{\alpha n}{4}$ edges incident to y have been deleted so far. Thus, we must have $|V_{\text{odd}}| \leq \frac{2}{\alpha}$.

Pair all remaining vertices of V_{odd} such that, in each pair, the vertices belong to a same connected component of G . By construction, $\delta(G) \geq \frac{3\alpha n}{4} - 1$. Thus, for each pair (x, y) in turn, we can find a path of length at most $\frac{5}{\alpha}$ between x and y , which we delete from G . Let \mathcal{P} be the set of edge-disjoint paths deleted. Note that $|E(\mathcal{P})| \leq \frac{5}{\alpha^2}$. Moreover, we have deleted at most $\ell + \frac{5}{\alpha^2} \leq n + \frac{\delta n}{2}$ edges in total. Finally, G is Eulerian and $\delta(G) \geq \frac{\alpha n}{2}$. Applying Theorem 1.10(ii) with $\frac{\alpha}{2}$ and $\frac{\delta}{2}$ playing the roles of α and δ completes the proof. \square

6.2. Some remarks on Theorem 1.11. As discussed in the introduction, we now show that neither the linear minimum degree condition (or even the stronger assumption of linear connectivity), nor the weakly- $(\frac{\alpha}{2}, p)$ -quasirandom property is sufficient on its own to obtain the bounds in Theorem 1.11.

Proposition 6.1. *For any odd integer $n \geq 20$, there exists an $\lfloor \frac{n}{10} \rfloor$ -connected Eulerian graph G on $2n$ vertices such that the following hold.*

- (i) G cannot be decomposed into fewer than $\max \left\{ \frac{\text{odd}(G)}{2}, \frac{\Delta(G)}{2} \right\} + \frac{n}{10}$ paths.
- (ii) G cannot be decomposed into fewer than $\frac{\Delta(G)}{2} + \frac{n}{10}$ cycles.

Proof. Assume G_1, G_2 are two vertex-disjoint cliques of size n and let $V_1 \subseteq V(G_1)$ and $V_2 \subseteq V(G_2)$ with $|V_1|, |V_2| = \lfloor \frac{n}{10} \rfloor$. Let G be obtained from $G_1 \cup G_2$ by adding two edge-disjoint perfect matchings between V_1 and V_2 . Note that G is an $\lfloor \frac{n}{10} \rfloor$ -connected Eulerian graph on $2n$ vertices with $\Delta(G) = n + 1$.

Since there are at most $\frac{n}{5}$ edges between G_1 and G_2 , any cycle decomposition of G will contain at most $\frac{n}{10}$ cycles with edges of both G_1 and G_2 and these will cover at most $\frac{n}{5}$ edges incident to each vertex of G . Thus any cycle decomposition of G will contain at least $\frac{4n+5}{10}$ cycles of G_1 and at least $\frac{4n+5}{10}$ cycles of G_2 . Therefore, any cycle decomposition of G will contain at least $\frac{4n}{5} + 1 > \frac{\Delta(G)}{2} + \frac{n}{10}$ cycles. Similar arguments show that G cannot be decomposed into fewer than $\frac{3n}{5} + 1 > \max \left\{ \frac{\text{odd}(G)}{2}, \frac{\Delta(G)}{2} \right\} + \frac{n}{10}$ paths. \square

Proposition 6.2. *For all $0 < \alpha \leq 1$, and all $n_0 \in \mathbb{N}^*$, the following hold.*

- (i) *There exists a weakly- $(\frac{\alpha}{2}, \frac{\alpha^2}{100})$ -quasirandom graph G on $n \geq n_0$ vertices such that G cannot be decomposed into fewer than $\max \left\{ \frac{\text{odd}(G)}{2}, \frac{\Delta(G)}{2} \right\} + \frac{\alpha n}{10}$ paths.*
- (ii) *There exists an Eulerian weakly- $(\frac{\alpha}{2}, \frac{\alpha^2}{100})$ -quasirandom graph G on $n \geq n_0$ vertices such that G cannot be decomposed into fewer than $\frac{\Delta(G)}{2} + \frac{\alpha n}{10}$ cycles.*

Proof. Let m be a sufficiently large odd integer, $\delta := \frac{\alpha}{10}$, and $\ell := \frac{2\delta m + 4}{1 - 2\delta}$.

For part (i), let S_ℓ be a star with ℓ leaves and K_m be a complete graph on m vertices such that $V(S_\ell) \cap V(K_m) = \{x\}$ for some leaf x of S_ℓ . Let $G := K_m \cup S_\ell$. Then G is graph of order $n := m + \ell$, with $\Delta(G) = m$ and at least ℓ vertices of odd degree. Let $A \cup B$ be a partition of $V(G)$ with $|A|, |B| \geq \frac{\alpha n}{2}$. Then, both A and B contain at least $\frac{\alpha n}{10}$ vertices of K_m . Thus, $e_G(A, B) \geq \frac{\alpha^2 n^2}{100} \geq \frac{\alpha^2}{100} |A| |B|$ and G is weakly- $(\frac{\alpha}{2}, \frac{\alpha^2}{100})$ -quasirandom. But, one can easily show that G cannot be decomposed into fewer than $\frac{m+1}{2} + \frac{\ell-2}{2} > \max \left\{ \frac{\text{odd}(G)}{2}, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\} + \frac{\alpha n}{10}$ paths.

For part (ii), let G be obtained from K_m by appending $\frac{\ell}{2}$ vertex-disjoint triangles with exactly one endpoint in $V(K_m)$. Clearly, G is an Eulerian graph on $n := m + \ell$ vertices with $\Delta(G) = m + 1$. Now let $A \cup B$ be a partition of $V(G)$ with $|A|, |B| \geq \frac{\alpha n}{2}$. Then, similarly as before, it follows that G is weakly- $(\frac{\alpha}{2}, \frac{\alpha^2}{100})$ -quasirandom. But G cannot be decomposed into fewer than $\frac{m-1}{2} + \frac{\ell}{2} > \frac{\Delta(G)}{2} + \frac{\alpha n}{10}$ cycles. \square

6.3. Some remarks on Conjecture 1.14. As discussed in the introduction, we show that the Erdős-Gallai conjecture is equivalent to Conjecture 1.14.

Proposition 6.3. *Conjecture 1.14 is equivalent to the Erdős-Gallai conjecture (Conjecture 1.4).*

Proof. (\Leftarrow) Assume Conjecture 1.4 holds and let c be a constant such that any N -vertex graph can be decomposed into at most cN cycles and edges, for each $N \in \mathbb{N}^*$. Let $\varepsilon \ll c^{-1}, p$. Let G be as in Conjecture 1.14 and $\mathcal{D} := \emptyset$. We repeatedly add cycles to \mathcal{D} until it forms a cycle decomposition of G . Weak- (ε, p) -quasirandomness implies that fewer than εn vertices of G have degree less than $\frac{pn}{2}$. Let S be the set of these vertices and apply the arguments of Step 1 of the proof Theorem 1.10(ii) with $G - S$ and $\frac{p}{2}$ playing the roles of G and α to obtain a decomposition of G into G^*, Γ, Γ' , and H . Add the vertices in S to the exceptional set V_0 and all edges incident to these vertices to G^* . Note that we now have $|V_0| \leq 2\varepsilon n$. Moreover, by similar arguments as in the proof of Theorem 1.11, the reduced graph R of G^* is connected.

Decompose $G[V_0]$ into at most $2c\varepsilon n$ cycles and edges. Add the cycles obtained to \mathcal{D} and delete their edges from G . By choosing a decomposition where the number of edges is minimal, we may assume $G[V_0]$ is now a forest and, thus, contains at most $2\varepsilon n$ edges.

Then, we can decompose $G[V_0]$ into $\ell \leq 2\varepsilon n$ edge-disjoint paths P_1, \dots, P_ℓ such that the following hold. For each $i \in [\ell]$, the endpoints x_i and y_i of P_i have odd degree in $G[V_0]$ and, moreover, each $x \in V_0$ is an endpoint of at most one of the P_i . Let $i \in [\ell]$. Since G is Eulerian, there exist $x'_i \in N_G(x_i) \setminus V_0$ and $y'_i \in N_G(y_i) \setminus V_0$. If $x'_i = y'_i$, add the cycle $x'_i x_i P_i y_i y'_i$ to \mathcal{D} . Otherwise, let $P'_i := x'_i x_i P_i y_i y'_i$.

Apply Lemma 4.15 with 2ε playing the role of ε and each \mathcal{P}_i consisting of exactly one of the paths P'_j constructed above. Add all the cycles obtained to \mathcal{D} and delete their edges from G^*, Γ, Γ' , and H . Thus, $G^*[V_0]$ is now empty. By Lemma 4.2 and part (b) of Lemma 4.15, V_0, V_1, \dots, V_k is now an $(\varepsilon^{\frac{1}{9}}, \beta, k, m, m', R')$ -superregular equalised partition of Γ and an $(\varepsilon^{\frac{1}{9}}, \zeta, k, m, m', R'')$ -superregular equalised partition of Γ' . Decompose the remainder of G as in Theorem 1.11(ii) (see Steps 3–5 of the proof of Theorem 1.10(ii)).

(\Rightarrow) Assume Conjecture 1.4 does not hold and assume for a contradiction that Conjecture 1.14 is true. Fix $\delta > 0$ and $\frac{1}{4} \geq p > 0$. Let $1 > \varepsilon > 0$ and n_0 be as in Conjecture 1.14 and fix a constant c such that $c \geq \delta(1 + \frac{2}{\varepsilon})$. Let H be an Eulerian graph on $m \geq \varepsilon n_0$ vertices such that any cycle decomposition of H contains more than cm cycles. Note that such graph exists since, as mentioned in the introduction, the Erdős-Gallai conjecture is equivalent to the problem of decomposing Eulerian graphs of order n in $O(n)$ cycles, and, by assumption, the Erdős-Gallai conjecture is false.

Assume without loss of generality that $\frac{2m}{\varepsilon}$ is an odd integer. Let G be the disjoint union of H and $K_{\frac{2m}{\varepsilon}}$. Note that G is a graph on $n = (1 + \frac{2}{\varepsilon})m \geq n_0$ vertices. Moreover, $\Delta(G) = \Delta(K_{\frac{2m}{\varepsilon}})$. Thus, any cycle decomposition of G will contain more than $\frac{\Delta(G)}{2} + cm \geq \frac{\Delta(G)}{2} + \delta n$ cycles. But, for any partition A, B of $V(G)$ with $|A|, |B| \geq \varepsilon n$, we have $|A|, |B| \geq \varepsilon(1 + \frac{2}{\varepsilon})m \geq 2m$ and thus $|A \cap V(K_{\frac{2m}{\varepsilon}})| \geq \frac{|A|}{2}$ and $|B \cap V(K_{\frac{2m}{\varepsilon}})| \geq \frac{|B|}{2}$. Therefore, $e_G(A, B) \geq \frac{1}{4}|A||B| \geq p|A||B|$ and G is weakly- (ε, p) -quasirandom, a contradiction. \square

Using Theorem 1.6 and the arguments of the proof of Proposition 6.3, one can show the following.

Proposition 6.4. *For any $\delta, p > 0$, there exist $\varepsilon, n_0 > 0$ such that the following hold. If G is an Eulerian weakly- $(\frac{\varepsilon}{\log \log n}, p)$ -quasirandom graph on $n \geq n_0$ vertices, then G can be decomposed into at most $\frac{\Delta(G)}{2} + \delta n$ cycles.*

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